Modeling Credit Valuation Adjustment and Wrong Way Risk

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ABSTRACT

After the credit crisis, credit valuation adjustment (CVA) has become a significant contributor to the daily P&L of trading books. As a result, it is crucial for banks to closely monitor the risk associated with CVA and set risk limit in place for a robust oversight and risk management framework. This paper presents a new approach for accurately calculating credit value adjustment (CVA). The model can achieve a high order of accuracy with a relatively easy implementation. Moreover, the model can naturally capture wrong or right way risk.

Key Words: credit value adjustment (CVA), wrong way risk, right way risk, credit risk modeling, least square Monte Carlo, derivative valuation, collateralization, margin and netting.

Following the global financial crisis, different adjustments have been introduced to the valuation of derivative contracts to account for counterparty and funding risk. Credit Valuation Adjustment (CVA) is an adjustment to the valuation of a portfolio in order to explicitly account for the credit worthiness of our counterparties. The CVA of an OTC derivatives portfolio with a given counterparty is the market value of the credit risk due to any failure to perform on agreements with that counterparty. This adjustment can be either positive or negative, depending on which of the two counterparties bears the larger burden to the other of exposure and of counterparty default likelihood.

CVA allows institutions not only to quantify counterparty risk as a single measurable P&L number, but also to dynamically manage, price and hedge counterparty risk. The benefits of CVA are widely acknowledged. Many banks have set up internal credit risk trading desks to manage counterparty risk on derivatives.

CVA, by definition, is the difference between the risk-free portfolio value and the risky portfolio value that takes into account the possibility of a counterparty's default. The risk-free portfolio value is what brokers quote or what trading systems or models normally report. The risky portfolio value, however, is a relatively less explored and less transparent area, which is the main challenge and core theme for CVA.

Previous CVA models involve the default time explicitly. Most CVA models in the literature (Brigo and Capponi (2008), Lipton and Sepp (2009), Pykhtin and Zhu (2006) and Gregory (2009), etc.) are based on this approach.

Although those models are very intuitive, they have the disadvantage that it explicitly involves the default time. We are very unlikely to have complete information about a firm's default point, which is often inaccessible (see Duffie and Huang (1996), Jarrow and Protter (2004), etc.). Usually, valuation under the DTA is performed via Monte Carlo simulation. On the other hand, however, the DPA relies on the probability distribution of the default time rather than the default time itself. Sometimes the DPA yields simple closed form solutions.

Joshi and Kwon (2016) propose a nested Monte Carlo simulation or least square technique to compute CVA. Graaf employ a finite-difference method and Monte Carlo simulation to solve a partial differential equation (PDE) and to estimate the mean exposure respectively. Borovykh et al. (2018) use a fast Fourier transform approach to calculate CVA.

This paper presents a framework for risky valuation and CVA. In contrast to previous studies, the model relies on the probability distribution of the default time rather than the default time itself.

After the credit crisis, a simplified assumption of independent default no longer holds especially when CVA is calculated against other financial institutions. The current framework should be further enhanced by embedding default correlations.

Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty. For example, in wrong way risk exposure tends to increase when counterparty credit quality worsens, while in right way risk exposure tends to decrease when counterparty credit quality declines. Wrong/right way risk, as an additional source of risk, is rightly of concern to banks and regulators. Since this new model allows us to incorporate correlated and potentially simultaneous defaults into risky valuation, it can naturally capture wrong/right way risk.

The rest of this paper is organized as follows: Section 2 discusses risky valuation and CVA. Section 2 presents numerical results. The conclusions are given in Section 3. All proofs and a practical framework that embraces netting agreements, margining agreements and wrong/right way risk are contained in the appendices.

1. Risky Valuation and CVA

CVA effectively computes the expected loss at default of the counterparty within maturity from the investor's point of view (conditional on no previous counterparty default). In fact, the CVA is defined as the negated difference of the value of a defaultable cashflow and the value of an analogous default-free cashflow.

The value CVA is composed of the expected loss due to the counterparty defaults (within maturity and before the investor default) minus the gain made by the investor in the event of its own defaults (before the counterparty default; conditional on no previous default for both investor and the counterparty).

The stopping (or default) time τ of a firm is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined as,

$$\tau = \inf\left\{ t : \int_0^t h(s, \Phi_s)\,ds \geq \Delta \right\} \tag{1}$$

where $h(t)$ or $h(t, \Phi_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Γ_t, and Δ is a unit exponential random variable independent of Φ_t.

The survival probability from time *t* to *s* in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u)\,du \right) \tag{2a}$$

The default probability for the period (*t*, *s*) in this framework is defined by

$$q(t,s) := P(\tau \leq s \mid \tau > t, Z) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u)\,du \right) \tag{2b}$$

Two counterparties are denoted as *A* and *B*. Let valuation date be *t*. Consider a financial contract that promises to pay a $X_T > 0$ from party *B* to party *A* at maturity date *T*, and nothing before date *T*. All calculations in the paper are from the perspective of party *A*. The risk free value of the financial contract is given by

$$V^F(t) = E\left[D(t,T) X_T \mid \mathcal{F}_t \right] \tag{3a}$$

where

$$D(t,T) = \exp\left[-\int_t^T r(u)du\right] \tag{3b}$$

where $E\{\bullet|\mathcal{F}_t\}$ denotes the expectation conditional on the $\mathcal{F}_t$, $D(t,T)$ denotes the risk-free discount factor at time t for the maturity T and $r(u)$ denotes the risk-free short rate at time u ($t \le u \le T$).

Next, we turn to risky valuation. In a unilateral credit risk case, we assume that party A is default-free and party B is defaultable.

If there has been no default before time T (i.e., $\tau > T$), the value of the contract at T is the payoff X_T. If a default happens before T (i.e., $t < \tau \le T$), a recovery payoff is made at the default time τ as a fraction of the market value given by $\varphi V(\tau)$ where φ is the default recovery rate and $V(\tau)$ is the market value at default. Under a risk-neutral measure, the value of this defaultable contract is the discounted expectation of all the payoffs and is given by

$$V(t) = E\left[\left(D(t,T)\, X_T 1_{\tau>T} + D(t,\tau)\varphi V(\tau)1_{\tau\le T}\right) | \mathcal{F}_t\right] \tag{4}$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise.

Valuation can also rely on the probability distribution of the default time rather than the default time itself. We divide the time period (t, T) into n very small time intervals (Δt) and assume that a default may occur only at the end of each very small period. In our derivation, we use the approximation $\exp(y) \approx 1 + y$ for very small y. The survival and the default probabilities for the period (t, $t + \Delta t$) are given by

$$\hat{p}(t) := p(t, t+\Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \tag{5a}$$

$$\hat{q}(t) := q(t, t+\Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \tag{5b}$$

The binomial default rule considers only two possible states: default or survival. For the one-period $(t, t+\Delta t)$ economy, at time $t + \Delta t$ the asset either defaults with the default probability

$q(t, t+\Delta t)$ or survives with the survival probability $p(t, t+\Delta t)$. The survival payoff is equal to the market value $V(t+\Delta t)$ and the default payoff is a fraction of the market value: $\varphi(t+\Delta t)V(t+\Delta t)$. Under a risk-neutral measure, the value of the asset at *t* is the expectation of all the payoffs discounted at the risk-free rate and is given by

$$V(t) = E\left\{\exp\left(-r(t)\Delta t\right)\left[\hat{p}(t) + \varphi(t)\hat{q}(t)\right]V(t+\Delta t)\middle|\mathcal{F}_t\right\} \approx E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \tag{6}$$

where $y(t) = r(t) + h(t)\left(1-\varphi(t)\right) = r(t) + c(t)$ denotes the risky rate and $c(t) = h(t)\left(1-\varphi(t)\right)$ is called the (short) credit spread.

Similarly, we have

$$V(t+\Delta t) = E\left\{\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right\} \tag{7}$$

Note that $\exp\left(-y(t)\Delta t\right)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t+\Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-y(t)\Delta t\right)E\left[\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right]\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\sum_{i=0}^{1} y(t+i\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_t\right\} \end{aligned} \tag{8}$$

By recursively deriving from *t* forward over *T* and taking the limit as Δt approaches zero, the risky value of the asset can be expressed as

$$V(t) = E\left\{\exp\left[-\int_t^T y(u)du\right]V(T)\middle|\mathcal{F}_t\right\} \tag{9}$$

We may think of $y(u)$ as the risk-adjusted short rate.

For a derivative contract, usually its payoff may be either an asset or a liability to each party. Thus, we further relax the assumption and suppose that X_T may be positive or negative.

In the case of $X_T > 0$, the survival value is equal to the payoff X_T and the default payoff is a fraction of the payoff φX_T. Whereas in the case of $X_T \le 0$, the contract value is the payoff itself, because the default risk of party B is irrelevant for unilateral risky valuation in this case. Therefore, we have

Proposition 1: *The unilateral risky value of the single-payment contract in a discrete-time setting is given by*

$$V(t) = E\left[F(t,T)X_T \middle| \mathcal{F}_t\right] \tag{10a}$$

where

$$F(t,T) = D(t,T)\left[1 - 1_{X_T \geq 0} q(t,T)\left(1 - \varphi(T)\right)\right] \tag{10b}$$

Proof: See the appendix.

Here $F(t,T)$ can be regarded as a risk-adjusted discount factor. Proposition 1 says that the unilateral risky valuation of the single payoff contract has a dependence on the sign of the payoff. If the payoff is positive, the risky value is equal to the risk-free value minus the discounted potential loss. Otherwise, the risky value is equal to the risk-free value.

Proposition 1 can be easily extended from one-period to multiple-periods. Suppose that a defaultable contract has m cash flows. Let the m cash flows be represented as $X_1, \ldots, X_m$ with payment dates $T_1, \ldots, T_m$. Each cash flow may be positive or negative. We have the following proposition.

Proposition 2: *The unilateral risky value of the multiple-payment contract is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{11a}$$

where $t = T_0$ *and*

$$F(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - 1_{(X_{j+1} + V(T_{j+1})) \geq 0} q(T_j, T_{j+1})\left(1 - \varphi(T_{j+1})\right)\right] \tag{11b}$$

Proof: See the appendix.

The CVA, by definition, can be expressed as

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[\left(D(t,T_i) - \prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{12}$$

Proposition 2 provides a general form for pricing a unilateral defaultable contract. Applying it to a particular situation in which we assume that all the payoffs are nonnegative, we derive the following corollary:

Corollary 1*: If all the payoffs are nonnegative, the risky value of the multiple-payments contract is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} \overline{F}(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{13a}$$

where $t = T_0$ *and*

$$\overline{F}(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - q(T_j, T_{j+1})\left(1 - \varphi(T_{j+1})\right)\right] \tag{13b}$$

The proof of this corollary is easily obtained according to Proposition 2 by setting $\left(X_{j+1} + V(T_{j+1})\right) \geq 0$, since the value of the contract at any time is also nonnegative.

The CVA in this case is given by

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[D(t, T_i)\left(1 - \prod_{j=0}^{i-1}\left(1 - q(T_j, T_{j+1})(1 - \varphi(T_{j+1}))\right)\right) X_i \middle| \mathcal{F}_t\right] \tag{14}$$

Given $\tau_2 > t$ ($\tau_0 > t$), it follows from [**Error! Reference source not found.**] that the unilateral CVA_0 from the investor's point of view (CVA_2) from the counterparty's point of view) is given by

$$CVA_0 = (1 - R_2)\mathbf{E}_t[1_{\{\tau_2 \leq T\}} A^{+}_{\tau_2}]; \tag{15}$$

$$CVA_2 = (1 - R_0)\mathbf{E}_t[1_{\{\tau_0 \leq T\}} (-A_{\tau_0})^{+}]; \tag{16}$$

The default of a firm's counterparty might affect its own default probability. Thus, default correlation and dependence arise due to the counterparty relations. Default correlation can be positive or negative. The effect of positive correlation is usually called contagion, whereas the latter is referred to as competition effect.

In theory, risk neutral default correlation should be given by parameter value calibrated from basket CDS. However, it is impossible to find price quotes as these are not market traded instruments.

The solution is to use proxy by correlation between credit spreads. Given the liquidity of 5-year spreads, correlation can be estimated based on historical spreads per rating and per sector. Alternatively, a weighted average of spread correlation across various terms can be used.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \tag{15a}$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \tag{15b}$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \tag{15c}$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \tag{15d}$$

where $E(Y_j) = q_j$, $\sigma_j^2 = p_j q_j$, $\sigma_{AB} := E[(Y_A - q_A)(Y_B - q_B)] = \rho_{AB}\sigma_A\sigma_B = \rho_{AB}\sqrt{q_A p_A q_B p_B}$ where ρ_{AB} denotes the default correlation coefficient and σ_{AB} denotes the default covariance.

Suppose that a financial contract that promises to pay a X_T from party *B* to party *A* at maturity date *T*, and nothing before date *T* where $T > t$. The payoff X_T may be positive or negative, i.e. the contract may be either an asset or a liability to each party. All calculations are from the perspective of party *A*.

At time *T*, there are a total of four ($2^2 = 4$) possible states shown in Table 1. The risky value of the contract is the discounted expectation of the payoffs and is given by the following proposition.

Proposition 3: *The bilateral risky value of the single-payment contract is given by*

$$V(t) = E\left[K(t,T) X_T \middle| \mathcal{F}_t\right] = E\left[D(t,T)\left(1_{X_T \geq 0} k_B(t,T) + 1_{X_T < 0} k_A(t,T)\right) X_T \middle| \mathcal{F}_t\right] \tag{16a}$$

where

$$k_B(t,T) = p_B(t,T)p_A(t,T) + \varphi_B(T)q_B(t,T)p_A(t,T) + \overline{\varphi}_B(T)p_B(t,T)q_A(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_B(T) - \overline{\varphi}_B(T) + \varphi_{AB}(T)\right) \quad (16b)$$

$$k_A(t,T) = p_B(t,T)p_A(t,T) + \varphi_A(T)q_A(t,T)p_B(t,T) + \overline{\varphi}_A(T)p_A(t,T)q_B(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_A(T) - \overline{\varphi}_A(T) + \varphi_{AB}(T)\right) \quad (16c)$$

Let □=□$_0$□□$_2$. Given □>t, it follows from [**Error! Reference source not found.**] that the bilateral CVA of the investor CVA_0 is given by:

$$CVA_0 = (1-R_2)\mathbf{E}_t[1_{\{\tau=\tau_2\leq T\}}A^+_{\tau_2}] - (1-R_0)E_t[1_{\{\tau=\tau_0\leq T\}}(-A_{\tau_0})^+]; \qquad (3)$$

The value CVA_0 is composed of the expected loss due to the counterparty defaults (within maturity and before the investor default) minus the gain made by the investor in the event of its own defaults (before the counterparty default; conditional on no previous default for both investor and the counterparty).

Let's compute the following

$$\mathbf{E}_t[1_{\{\tau=\tau_2\leq T\}}A^+_{\tau_2}] = \mathbf{E}_t[1_{\{\tau_0\geq\tau_2\leq T\}}A^+_{\tau_2}]$$

$$= \int_0^T \mathbf{E}_t[1_{\{\tau_0\geq u\}}A^+_{\tau_2}|\tau_2=u]dP(\tau_2\leq u)$$

Under the assumption of independence of τ_2 and τ_0 the expectation can be expressed as

$$= \int_0^T P(\tau_0\geq u)\mathbf{E}_t[A^+_{\tau_2}|\tau_2=u]dP(\tau_2\leq u)$$

Under the assumption that $\mathbf{E}_t[A^+_{\tau_2}|\tau_2=u]=\mathbf{E}_t[A^+_u]$.,, the expectation can be expressed as

$$= \int_0^T P(\tau_0 \geq u)\mathbf{E}_t[A_u^+]dP(\tau_2 \leq u)$$

The integral can be approximated using numerical integration.

$$= \sum_i P(\tau_0 \geq T_i)[P(\tau_2 \leq T_i) - P(\tau_2 \leq T_{i-1})]\mathbf{E}_t[A_{T_i}^+]$$

Another way to look at it is as follows. Let us partition the $[0,T]$ interval to $T_0 = t, T_1, T_2, \ldots, T_n = T$, then we have

$$\begin{aligned} \mathbf{E}_t[1_{\{\tau=\tau_2 \leq T\}} A_{\tau_2}^+] &= \mathbf{E}_t[1_{\{\tau_0 \geq \tau_2 \leq T\}} A_{\tau_2}^+] \\ &= \sum_i \mathbf{E}_t[1_{\{\tau_0 \geq \tau_2, T_{i-1} < \tau_2 \leq T_i\}} A_{\tau_2}^+] \end{aligned}$$

2. Numerical Results

In this section, we present some numerical results for CVA calculation based on the theory described above. First, we study the impact of margin agreements on CVA. The testing portfolio consists of a number of interest rate and equity derivatives. The number of simulation scenarios (or paths) is 20,000. The time buckets are set weekly. If the computational requirements exceed the system limit, one can reduce both the number of scenarios and the number of time buckets. The time buckets can be designed fine-granularity at the short end (e.g., daily and then weekly) and coarse-granularity at the far end (e.g. monthly and then yearly). The rationale is that the calculation becomes less accurate due to the accumulated error from simulation discretization, and inherited errors from calibration of the underlying models, such as those due to the change of macro-economic climate. The collateral margin period of risk is assumed to be 14 days (2 weeks).

For risk-neutral simulation, we use a Hull-White model for interest rate and a CIR (Cox-Ingersoll-Ross) model for hazard rate scenario generations a modified GBM (Geometric Brownian Motion) model for equity and collateral evolution. The results are presented in the following tables. Table 2 illustrates that if party *A* has an infinite collateral threshold $H_A = \infty$ i.e., no collateral requirement on *A*, the CVA value increases while the threshold H_B increases. Table 3 shows that if party *B* has an infinite collateral threshold $H_B = \infty$, the CVA value actually decreases while the threshold H_A increases. This reflects the bilateral impact of the collaterals on the CVA. The impact is mixed in Table 4 when both parties have finite collateral thresholds.

Table 2. The impact of collateral threshold H_B on the CVA

This table shows that given an infinite H_A, the CVA increases while H_B increases, where H_B denotes the collateral threshold of party *B* and H_A denotes the collateral threshold of party *A*.

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	19,550.91	20,528.65	21,368.44	22,059.30

Table 3. The impact of collateral threshold H_A on the CVA

This table shows that given an infinite H_B, the CVA decreases while H_A increases, where H_B denotes the collateral threshold of party *B* and H_A denotes the collateral threshold of party *A*.

Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	28,283.64	25,608.92	23,979.11	22,059.30

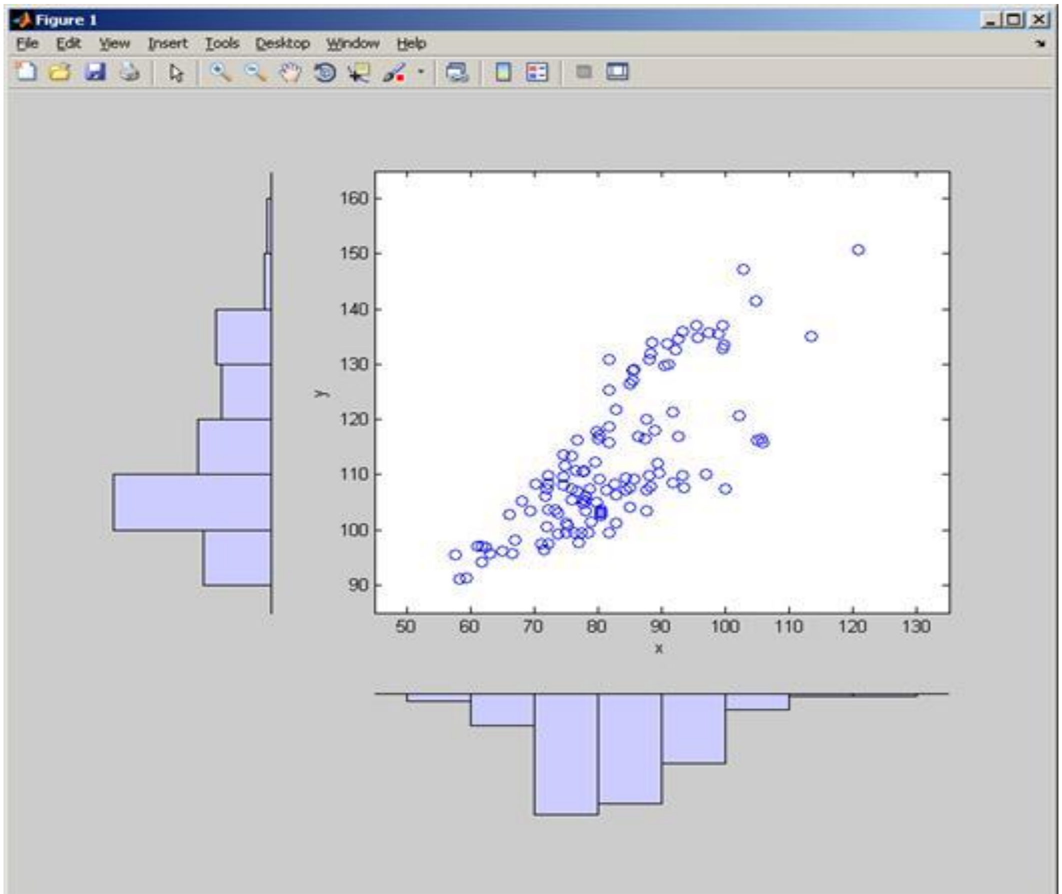
Table 4. The impact of the both collateral thresholds on the CVA

The CVA may increase or decrease while both collateral thresholds change, where H_B denotes the collateral threshold of party *B* and H_A denotes the collateral threshold of party *A*. This reflects the fact that the collaterals have bilateral impacts on the CVA.

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	25,752.98	22,448.45	23,288.24	22,059.30

Next, we examine the impact of wrong way risk. Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty. Wrong/right way risk, as an additional source of risk, is rightly of concern to banks and regulators.

Some financial markets are closely interlinked, while others are not. For example, CDS price movements have a feedback effect on the equity market, as a trading strategy commonly employed by banks and other market participants consists of selling a CDS on a reference entity and hedging the resulting credit exposure by shorting the stock. On the other hand, Moody's Investor's Service (2000) presents statistics that suggest that the correlations between interest rates and CDS spreads are very small. The shape of 5-year credit spread is shown below.



To capture wrong/right way risk, we need to determine the dependency between counterparties and to correlate the credit spreads or hazard rates with the other market risk factors, e.g. equities, commodities, etc., in the scenario generation.

We use an equity swap as an example. Assume the correlation between the underlying equity price and the credit quality (hazard rate) of party B is ρ . The impact of the correlation on the CVA is show in Table 5. The results say that the CVA increases when the absolute value of the negative correlation increases.

Table 5. The impact of wrong way risk on the CVA

This table shows that the CVA increases while the negative correlation ρ increases in the absolute value. We use an equity swap as an example and assume that there is a negative correlation between the equity price and the credit quality of party *B*.

Correlation ρ	0	-50%	-100%
CVA	165.15	205.95	236.99

As we increase default correlation, we expect a monotonically decreasing trend in EL (for both normal and reverse views). This is because as the default correlation increases and given one party defaults, the survival probability of the other party would decrease.

3. Conclusion

The occurrence of CVA is linked to the recent financial crisis. Financial institutions used to disregard, or at least minimize, credit and funding costs. Historically, the funding for taking on in-the-money trades and for posting collateral could be achieved through a variety of low-cost funding options and this low cost was to some extent offset by the interest paid on collateral by the receiving party.

The financial crisis has changed this view as funding has become more costly, and the interest paid on collateral no longer offsets the increased cost. The evaluation and controlling of funding costs have therefore become a critical part of the risk management strategy of financial institutions.

This article presents a framework for pricing risky contracts and their CVAs. The model relies on the probability distribution of the default jump rather than the default jump itself, because the default jump is normally inaccessible. We find that the valuation of risky assets and their CVAs, in most situations, has a backward recursive nature and requires a backward induction valuation. An intuitive explanation is that two counterparties implicitly sell each other an option to default when entering into an OTC derivative transaction. If we assume that a default may occur at any time, the default options are American style options. If we assume that a default may only happen on the payment dates, the default options are Bermudan style options. Both Bermudan and American options require backward induction valuations.

Based on our theory, we propose a novel cash-flow-based framework (see appendix) for calculating bilateral CVA at the counterparty portfolio level. This framework can easily incorporate various credit mitigation techniques, such as netting agreements and margin agreements, and can capture wrong/right way risk. Numerical results show that these credit mitigation techniques and wrong/right way risk have significant impacts on CVA.

Appendix

A practical framework for calculating bilateral CVA

We developed a practical framework for calculating bilateral CVA at counterparty portfolio level based on the theory described above. The framework incorporates netting and margin agreements, and captures right/wrong way risk.

Two parties are denoted as *A* and *B*. All calculations are from the perspective of party *A*. Let the valuation date be *t*. The CVA computation procedure consists of the following steps.

B.1. Risk-neutral Monte Carlo scenario generation

One core element of the trading credit risk modeling is the Monte Carlo scenario generation (market evolution). This must be able to run a large number of scenarios for each risk factor with flexibility over parameterization of processes and treatment of correlation between underlying factors. Credit exposure may be calculated under real probability measure, while CVA or pricing counterparty credit risk should be conducted under risk-neutral probability measure.

Due to the extensive computational intensity of pricing counterparty risk, there will inevitably be some compromise of limiting the number of market scenarios (paths) and the number of simulation dates (also called "time buckets" or "time nodes"). The time buckets are normally designed fine-granularity at the short end and coarse-granularity at the far end. The details of scenario generation are beyond the scope of this paper.

A.2. Cash flow generation

For ease of illustration, we choose a vanilla interest rate swap, as interest rate swaps collectively account for around two-thirds of both the notional and market value of all outstanding derivatives.

Assume that party *A* pays a fixed rate, while party *B* pays a floating-rate. Assume that there are *M* time buckets ($T_0, T_1, ..., T_M$) in each scenario and *N* cash flows in the sample swap. Let consider scenario *j* first.

For swaplet *i*, there are four important dates: the fixing date $t_{i,f}$, the starting date $t_{i,s}$, the ending date $t_{i,e}$ and the payment date $t_{i,p}$. In general, these dates are not coincidently at the simulation time buckets. The time relationship between swaplet *i* and the simulation time buckets is illustrated in Figure B1.

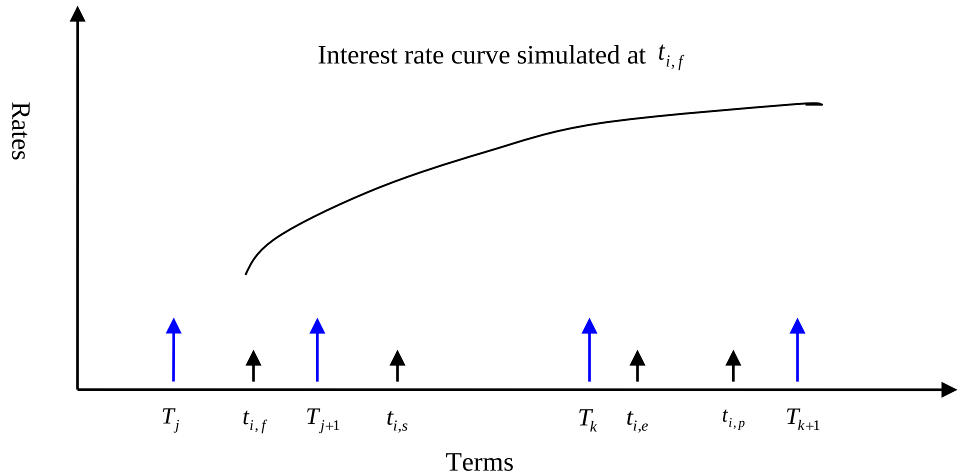


Figure B1: An interest rate swaplet

This figure illustrates the time relationship between an interest rate swaplet and the simulation time buckets. The floating leg of the swaplet is reset at the fixing date $t_{i,f}$ with the starting date $t_{i,s}$, the ending date $t_{i,e}$, and the payment date $t_{i,p}$. The simulation time buckets are $T_i, T_{i+1}, ..., T_{k+1}$. The simulated interest rate curve is starting at $t_{i,f}$. Both fixed rate payments and floating-rate payments occur on the same payment dates.

The cash flow of swaplet i is determined at the fixing date $t_{i,f}$ that is assumed to be between the simulation time buckets T_j and T_{j+1}. First, we need to create an interest rate curve (see https://finpricing.com/lib/IrCurveIntroduction.html) observed at $t_{i,f}$ by interpolating the interest rate curves simulated at T_j and T_{j+1} via either Brownian Bridge or linear interpolation. The linear interpolation is the expectation of the Brownian Bridge. Then we can calculate the payoff of swaplet i at scenario j as

$$\chi_{j,i} = N\left(F(t_{i,f}; t_{i,s}, t_{i,e}) - R\right)\delta(t_{i,s}, t_{i,e}) \tag{B1}$$

where N denotes the notional; $F(t_{i,f};t_{i,s},t_{i,e})$ denotes the simply compounded forward rate reset at $t_{i,f}$ for the forward period ($t_{i,s}$, $t_{i,e}$); $\delta(t_{i,s},t_{i,e})$ denotes the accrual factor or day count fraction for the period ($t_{i,s}$, $t_{i,e}$) and R denotes the fixed rate.

The cash flow amount calculated by (B1) is paid on the payment date $t_{i,p}$. This value should be allocated into *the nearest previous time bucket* T_k as:

$$\tilde{\chi}_{j,k,i} = \chi_{j,i} D(T_k, t_{i,p}) \tag{B2}$$

where $D(T_k, t_{i,p})$ denotes the risk-free discount factor based on the interest rate curve simulated at T_k .

Cash flow generation for products without early-exercise provision is quite straightforward. For early-exercise products, one can use the approach proposed by Longstaff and Schwartz (2001) to obtain the optimal exercise boundaries and then the payoffs.

B3. Aggregation and netting agreements

When a given portfolio does not allow for cross-product netting, the portfolio level CVA is given by the sum of the individual sub-portfolio's stand alone CVAs. When netting and margin agreements are in place and the expected exposure of a counterparty is calculated depending on the netting and collateral agreements.

After generating cash flows for each deal, we need to aggregate them at counterparty portfolio level at each scenario and each time bucket. The cash flows are aggregated by either netting or nonnetting based on the netting agreements.

For netting, we add all cash flows together at the same scenario and the same time bucket to recognize offsetting. The aggregated cash flow under netting at scenario j and time bucket k is given by

$$\tilde{\chi}_{j,k} = \sum_i \tilde{\chi}_{j,k,i} \tag{B3}$$

For nonnetting, we divided cash flows into positive and negative groups and add them separately. In other words, the offsetting is not recognized. The aggregated cash flows under nonnetting at scenario *j* and time bucket *k* are given by

$$\tilde{\chi}_{j,k} = \begin{cases} \sum_l \tilde{\chi}_{j,k,l} & if \ \chi_{j,k,l} \geq 0 \\ \sum_m \tilde{\chi}_{j,k,m} & if \ \chi_{j,k,m} < 0 \end{cases} \tag{B4}$$

B4. CVA Calculation

After aggregating all cash flows via netting, one can price a portfolio in the same manner as pricing a single deal. We assume that the reader is familiar with the least square Monte Carlo valuation model proposed by Longstaff and Schwartz (2001) and thus do not repeat some well-known procedures for brevity.

CVA is by definition the difference between the risk-free portfolio value and the true (or risky or defaultable) portfolio value.

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