Lecture 4 - Atoms in oscillating fields

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Abstract

In the lecture, we will see how a time dependent coupling allows us to engineer a new Hamiltonian. Most importantly, we will discuss the resonant coupling of two levels and the decay of a single level to a continuum.

In the last lecture (Jendrzejewski et al.), we discussed the properties of two coupled levels. However, we did not elaborate at any stage how such a system might emerge in a true atom. Two fundamental questions come to mind:

1. How is it that a laser allows to treat two atomic levels of very different energies as if they were degenerate ?
2. An atom has many energy levels E_n and most of them are not degenerate. How can we reduce this complicated structure to a two-level system?

The solution is to resonantly couple two of the atom's levels by applying an external, oscillatory field, which is very nicely discussed in chapter 12 of Ref. (Jean-Louis Basdevant, 2002) (Cohen-Tannoudji et al., 1998). We will discuss important and fundamental properties of systems with a time-dependent Hamiltonian.

We will discuss a simple model for the atom in the oscillatory field. We can write down the Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{V}(t). \tag{1}$$

Here, $\hat{H}_0$ belongs to the atom and $V(t)$ describes the time-dependent field and its interaction with the atom. We assume that $|n\rangle$ is an eigenstate of $\hat{H}_0$ and write:

$$\hat{H}_0 |n\rangle = E_n |n\rangle . \tag{2}$$

If the system is initially prepared in the state $|i\rangle$, so that

$$|\psi(t=0)\rangle = |i\rangle , \tag{3}$$

what is the probability

$$P_m(t) = |\langle m|\psi(t)\rangle|^2 \tag{4}$$

to find the system in the state $|m\rangle$ at the time t?

1 Evolution Equation

The system $|\psi(t)\rangle$ can be expressed as follows:

$$|\psi(t)\rangle = \sum_n \gamma_n(t) \mathrm{e}^{-iE_n t/\hbar} |n\rangle \,, \tag{5}$$

where the exponential is the time evolution for $\hat{H}_1 = 0$. We plug this equation in the Schrödinger equation and get:

$$i\hbar \sum_n \left(\dot{\gamma}_n(t) - i\frac{E_n}{\hbar}\gamma_n(t)\right) \mathrm{e}^{-iE_n t/\hbar} |n\rangle = \sum_n \gamma_n(t) \mathrm{e}^{-iE_n t/\hbar} \left(\hat{H}_0 + \hat{V}\right) |n\rangle \tag{6}$$

$$\Longleftrightarrow i\hbar \sum_n \dot{\gamma}_n(t) \mathrm{e}^{-iE_n t/\hbar} |n\rangle = \sum_n \gamma_n(t) \mathrm{e}^{-iE_n t/\hbar} \hat{V} |n\rangle \tag{7}$$

If we multiply (6) with $\langle k|$ we obtain a set of coupled differential equations

$$i\hbar\dot{\gamma}_k \mathrm{e}^{-iE_k t/\hbar} = \sum_n \gamma_n \mathrm{e}^{-E_n t/\hbar} \langle k| \hat{V} |n\rangle \,, \tag{8}$$

$$i\hbar\dot{\gamma}_k = \sum_n \gamma_n \mathrm{e}^{-i(E_n - E_k)t/\hbar} \langle k| \hat{V} |n\rangle \tag{9}$$

with initial conditions $|\psi(t=0)\rangle$. They determine the full time evolution.

The solution of this set of equations depends on the details of the system. However, there are a few important points:

- For short enough times, the dynamics are driving by the coupling strength $\langle k| \hat{V} |n\rangle$.
- The right-hand sight will oscillate on time scales of $E_n - E_k$ and typically average to zero for long times.
- If the coupling element is an oscillating field $\propto e^{i\omega_L t}$, it might put certain times on resonance and allow us to avoid the averaging effect. It is exactly this effect, which allows us to isolate specific transitions to a very high degree [1]

We will now see how the two-state system emerges from these approximations and then set-up the perturbative treatment step-by-step.

2 Rotating wave approximation

We will now assume that the coupling term in indeed an oscillating field with frequency ω_L, so it reads:

$$\hat{V} = \hat{V}_0 \cos(\omega_L t) = \frac{\hat{V}_0}{2}\left(e^{i\omega_l t} + e^{-i\omega_l t}\right) \tag{10}$$

We will further assume the we would like use it to isolate the transition $i \to f$, which is of frequency $\hbar\omega_0 = E_f - E_i$. The relevant quantity is then the detuning $\delta = \omega_0 - \omega_L$. If it is much smaller than any other energy difference $E_n - E_i$, we directly reduce the system to the following closed system:

$$i\dot{\gamma}_i = \gamma_f \mathrm{e}^{-i\delta t}\Omega \tag{11}$$

$$i\dot{\gamma}_f = \gamma_i \mathrm{e}^{i\delta t}\Omega^* \tag{12}$$

[1] This is the idea behind atomic and optical clocks, which work nowadays at 10^{-18}.

Here we defined $\Omega = \langle i| \frac{\hat{V}_0}{2\hbar} |f\rangle$. And to make it really a time-of the same form as the two-level system from the last lecture, we perform the transformation $\gamma_f = \tilde{\gamma}_f e^{i\delta t}$, which reduces the system too:

$$i\dot{\gamma}_i = \Omega\tilde{\gamma}_f \tag{13}$$

$$i\dot{\tilde{\gamma}}_f = \delta\tilde{\gamma}_f + \Omega^*\gamma_i \tag{14}$$

This has exactly the form of the two-level system that we studied previously.

2.1 Adiabatic elimination

We can now proceed to the quite important case of far detuning, where $\delta \gg \Omega$. In this case, the final state $|f\rangle$ gets barely populated and the time evolution can be approximated to to be zero (luk).

$$\dot{\tilde{\gamma}}_f = 0 \tag{15}$$

We can use this equation to eliminate γ from the time evolution of the ground state. This approximation is known as *adiabatic elimination*:

$$\tilde{\gamma}_f = \frac{\Omega^*}{\delta}\gamma_i \tag{16}$$

$$\Rightarrow i\hbar\dot{\gamma}_i = \frac{|\Omega|^2}{\delta}\tilde{\gamma}_i \tag{17}$$

The last equation described the evolution of the initial state with an energy $E_i = \frac{|\Omega|^2}{\delta}$. If the Rabi coupling is created through an oscillating electric field, i.e. a laser, this is know as the **light shift** or the **optical dipole potential**. It is this concept that underlies the optical tweezer for which Arthur Ashkin got the nobel prize in the 2018 (201).

2.2 Example: Atomic clocks in optical tweezers

A neat example that ties the previous concepts together is the recent paper (rea). The experimental setup is visualized in Fig. 1.

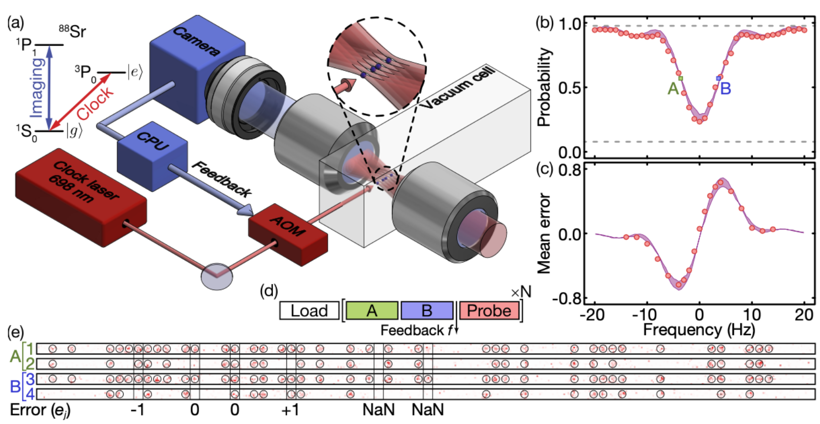

Figure 1: Experimental setup of an atomic array optical clock as taken from (rea).

While nice examples these clocks are still far away from the best clocks out there, which are based on optical lattice clocks and ions (Ludlow et al., 2015).

3 Perturbative Solution

The more formal student might wonder at which points all these rather hefty approximation are actually valid, which is obviously a very substantial question. So, we will now try to isolate the most important contributions to the complicated system through perturbation theory. For that we will assume that we can write:

$$\hat{V}(t) = \lambda \hat{H}_1(t) \tag{18}$$

, where λ is a small parameter. In other words we assume that the initial system $\hat{H}_0$ is only weakly perturbed. Having identified the small parameter λ, we make the *perturbative ansatz*

$$\gamma_n(t) = \gamma_n^{(0)} + \lambda\gamma_n^{(1)} + \lambda^2\gamma_n^{(2)} + \cdots \tag{19}$$

and plug this ansatz in the evolution equations and sort them by terms of equal power in λ.

The 0th order reads

$$i\hbar\dot{\gamma}_k^{(0)} = 0. \tag{20}$$

The 0th order does not have a time evolution since we prepared it in an eigenstate of $\hat{H}_0$. Any evolution arises due the coupling, which is at least of order λ.

So, for the 1st order we get

$$i\hbar\dot{\gamma}_k^{(1)} = \sum_n \gamma_n^{(0)} \mathrm{e}^{-i(E_n - E_k)t/\hbar} \langle k| \hat{H}_1 |n\rangle \,. \tag{21}$$

3.1 First Order Solution (Born Approximation)

For the initial conditions $\psi(t=0) = |i\rangle$ we get

$$\gamma_k^{(0)}(t) = \delta_{ik}. \tag{22}$$

We plug this in the 1st order approximation (21) and obtain the rate for the system to go to the final state $|f\rangle$:

$$i\hbar\dot{\gamma}^{(1)} = \mathrm{e}^{i(E_f - E_i)t/\hbar} \langle f| \hat{H}_1 |i\rangle \tag{23}$$

Integration with $\gamma_f^{(1)}(t=0) = 0$ yields

$$\gamma_f^{(1)} = \frac{1}{i\hbar} \int_0^t \mathrm{e}^{i(E_f - E_i)t'/\hbar} \langle f| \hat{H}_1(t') |i\rangle \, \mathrm{d}t', \tag{24}$$

so that we obtain the probability for ending up in the final state:

$$P_{i\to f}(t) = \lambda^2 \left|\gamma_f^{(1)}(t)\right|^2. \tag{25}$$

Note that $P_{i\to f}(t) \ll 1$ is the condition for this approximation to be valid!

Example 1: Constant Perturbation.

We apply a constant perturbation in the time interval $[0, T]$, as shown in 2. If we use (24) and set $\hbar\omega_0 = E_f - E_i$, we get

$$\gamma_f^{(1)}(t \geq T) = \frac{1}{i\hbar} \langle f| \hat{H}_1 |i\rangle \frac{\mathrm{e}^{i\omega_0 T} - 1}{i\omega_0}, \tag{26}$$

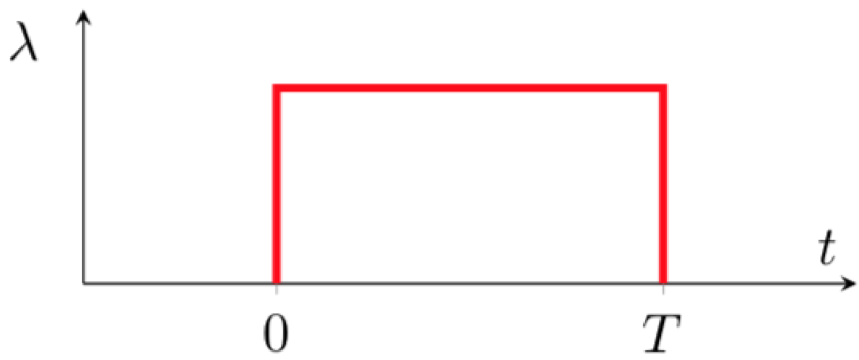

Figure 2: Sketch of a constant perturbation

and therefore

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \underbrace{\frac{\sin^2\left(\omega_0\frac{T}{2}\right)}{\left(\frac{\omega_0}{2}\right)^2}}_{\mathrm{y}(\omega_0,T)}. \tag{27}$$

A sketch of $\mathrm{y}(\omega_0, T)$ is shown in 3.

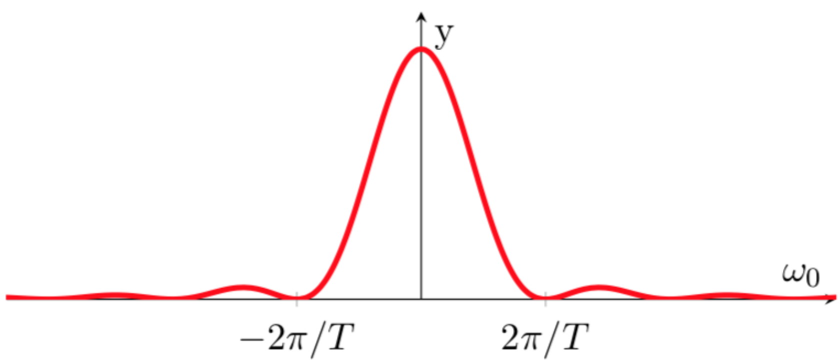

Figure 3: A sketch of y

We can push this calculation to the extreme case of $T \to \infty$. This results in a delta function, which is peaked round $\omega_0 = 0$ and we can write:

$$P_{i\to f} = T\frac{2\pi}{\hbar^2}\left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \delta(\omega_0) \tag{28}$$

This is the celebrated **Fermi's golden rule**.

Example 2: Sinusoidal Perturbation. For the perturbation

$$\hat{H}_1(t) = \begin{cases} \hat{H}_1\mathrm{e}^{-i\omega t} & \text{for } 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

we obtain the probability

$$P_{i\to f}(t \geq T) = \frac{1}{\hbar^2}\left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \mathrm{y}(\omega_0 - \omega, T). \tag{30}$$

At $\omega = |E_f - E_i| / \hbar$ we are on resonance.

In the fifth lecture, we will start to dive into the hydrogen atom.

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