Approximating the Riemann Zeta Function with Coprime n-tuples

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Abstract. At the age of twenty-eight, Leonhard Euler, now one of the greatest mathematicians of all time, gained immediate fame after solving the Basel problem. For over eighty years this problem had withstood the attempts of leading mathematicians until it was finally solved by Euler in 1734. Euler generalized the problem and over a century later Bernhard Riemann defined his zeta function, extending the problem to complex numbers. Riemann's zeta function has since become the head of the most important unsolved problem in pure mathematics, the Riemann hypothesis.

1. Introduction

The aforementioned Basel problem, which went unsolved for over eighty years can be approximated with a simple experiment.
First you ask a number of people to each list two random (preferably big) integers. Then count how many of the two number pairs are coprime. (A group of numbers is coprime if the only positive integer that is a shared divisor is 1.)
The total number of pairs divided by the number of coprime pairs will be approximately the solution to the Basel problem, $\pi^2/6$.
To rephrase the result, the probability of two random integers being coprime is $6/\pi^2$, the inverse of the solution.

2. Basel Problem

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers.
Defined as an infinite series, the problem looks like:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

The sum of this series is approximately equal to 1.644934. The Basel problem asks for the exact sum of the series (in closed form) and a proof, so the decimal approximation isn't good enough. In 1734 Euler found the exact sum to be $\pi^2/6$. Euler's proof was flawed in that he used math that hadn't been justified at the time, but he produced an accepted proof just 7 years later. His original proof was later found justified.

3. Euler's Solution

We will now solve the problem in the way Euler did.
To begin, recall the Taylor series of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Also recall the product definition for $\sin \pi x$:

$$\sin(\pi x) = \pi x \left(1 - x^2\right)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right)\cdots$$

formally multiply the product definition for $\sin \pi x$

$$= \pi x$$

$$-\pi x^3 \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right]$$

$$+\pi x^5 \left[\frac{1}{1 \cdot 4} + \frac{1}{1 \cdot 9} + \cdots + \frac{1}{4 \cdot 9} + \cdots\right]$$

$$- \cdots$$

Notice that doing this leaves separate terms for $x, x^3, x^5 \ldots$
Now use the Taylor series of $\sin x$ referenced above and plug in πx for x

$$\sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots$$

Simplify

$$= \pi x - \frac{\pi^3}{6} x^3 + \frac{\pi^5}{120} x^5 - \cdots$$

You will notice that the Taylor series for $\sin \pi x$ also leaves separate terms for $x, x^3, x^5 \ldots$
You can set the x^3 term from the expanded product and the x^3 term from the substituted Taylor series equal to each other (you can do this because of Newton's Identities)

$$-\pi \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right] = -\frac{\pi^3}{6}$$

You will notice that the equation between the brackets is actually the Basel problem. Substitute the Basel problem in for its Taylor series and you will get:

$$-\pi \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^3}{6}$$

Divide both sides by $-\pi$ and you are left with the solution to the Basel problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

4. Proving the Experiment

In the experiment in the introduction, we found that the probability of two random integers being coprime was approximately $6/\pi^2$, the inverse of the solution to the Basel problem. We will now prove why we get this result.

To do this we will use Euler's totient function $\varphi(n)$. Euler's totient function counts the positive integers up to a given integer n that are relatively prime (coprime) to n. In other words, it is the number of integers k in the range $1 \leq k \leq n$ for which the greatest common divisor $\gcd(n, k)$ is equal to 1. For ease of use we can write this as a summation:

$$\varphi(n) = \sum_{k=1}^{n} \{\gcd(n, k) = 1\}$$

Now that we have the totient function defined, we can define the result of our experiment as the limit of a double summation:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \{\gcd(m, n) = 1\} \to \frac{6}{\pi^2}$$

This double summation goes over every single integer m and n up to N, counts how many pairs (m, n) are coprime and divides that number by the total number of pairs checked (N^2). as N approaches infinity, the double summation approaches $\frac{6}{\pi^2}$. Using a summation identity you can simplify this to:

$$\lim_{N\to\infty} \frac{2}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{n-1} \{\gcd(m, n) = 1\} \to \frac{6}{\pi^2}$$

You can then substitute the second summation for Euler's totient function:

$$\lim_{N\to\infty} \frac{2}{N^2} \sum_{n=1}^{N} \varphi(n) \to \frac{6}{\pi^2}$$

Shown by Arnold Walfisz, the average order of the totient function is given by:

$$\sum_{n<x} \varphi(n) = \frac{3}{\pi^2} x^2 + O\left(x (\log x)^{2/3} (\log\log x)^{4/3}\right)$$

As we are working with the limit as the function approaches infinity, we can remove the error term O and plug the average order -$\sum \varphi(n)$- into our summation. This leaves us with:

$$\lim_{N\to\infty} \frac{2}{N^2} \cdot \frac{3}{\pi^2} N^2 \to \frac{6}{\pi^2}$$

Multiply out the fractions, removing the N^2 term, in turn removing the need for the limit. We are now left with our solution:

$$\frac{6}{\pi^2} = \frac{6}{\pi^2}$$

5. Extending the Experiment

In his 1859 paper "On the Number of Primes Less Than a Given Magnitude", Bernhard Riemann defined his zeta function, expanding upon the series from the

Basel problem, as the following:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

You may notice that $\zeta(2)$ is actually the Basel problem.

Riemann's Zeta function is actually defined over complex numbers, but in our case we will only be defining it over positive integers.

Now that we have proven that the probability of 2 random integers being coprime is equal to the inverse of $\zeta(2)$, we can actually extend the experiment for s random integers.

In 1972, James E. Nymann showed that k integers, chosen independently and uniformly from $\{1, \cdots, n\}$, are coprime with probability $\frac{1}{\zeta(k)}$ as n goes to infinity, where ζ refers to the Riemann zeta function.

I'm not even going to attempt to explain his proof, I will instead show code that will let you run the experiment for any amount of numbers. The code is written in Python.

```
from random import randint
from math import gcd
from sys import maxsize

k = 5 # tuple length
samples = 100000 # number of random tuples tested

cptuples = 0 # count of coprime tuples

for i in range(samples):
  nums = tuple(randint(1,maxsize) for i in range(k))
  if gcd(*nums) == 1: cptuples += 1
print(cptuples/samples) # 1/zeta(k)
```

6. Conclusion

A simple experiment can be used to approximate the results of a complicated problem.

You can use the probability of an n-tuple being coprime to approximate Zeta(n)

References

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