Conservation Laws and Explicit Solution of system of Fractional-Order Coupled Nonlinear Hirota Equations by Lie Symmetry Analysis

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Abstract

The main objective of this research article is to summarize the study of the application of Lie symmetry reduction to the fractional-order coupled nonlinear complex Hirota system of partial differential equations. By the efficient use of symmetries and explicit solutions, this system reducing to nonlinear fractional ordinary differential equations (FODEs) with the application of Erdyli-Kober (E-K) operators for fractional derivatives and integrals depending on real order. Investigating the convergent series solution along with adjoint system and providing the conservation laws by Noether's theorem.

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Abstract: The main objective of this research article is to summarize the study of the application of Lie symmetry reduction to the fractional order coupled nonlinear complex Hirota system of partial differential equations. By the efficient use of symmetries and explicit solutions, this system reducing to nonlinear fractional ordinary differential equations (FODEs) with the application of Erdyli-Kober (E-K) operators for fractional derivatives and integrals depending on real order θ. Investigating the convergent series solution along with the adjoint system and providing the conservation laws by Noether's theorem.

Keywords: Lie symmetry reduction, Nonlinear system of Hirota equations, Erdyli-Kober operators, Noether's theorem.

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1. Introduction

Fractional calculus is the theory of fractional integrals and derivatives of arbitrary order which is evolved towards the end of 17th century. Evidently, fractional calculus is old as classical calculus; many physicists and mathematicians [1, 3-5] like Lagranges, Euler, Abel, Gronwald, Letnikov, Riemann and Caputo have done great contribution on fractional-order derivatives. In the current research, there have been fast development in the field of fractional order systems involving in science, engineering and technology i.e. bioscience, fluid flow, electromagnetic, viscoelasticity, pandemic analysis, medicine, infectious modeling, drug therapy, image processing and diffusion wave equations [9, 10, 15, 16, 32].

Every researcher or mathematician has open challenge to overcome the difficulties and to find the exact solution of fractional-order systems. The person, who knows the utility of fractional derivatives and integrals, is only authorized to work on such kind of fractional partial differential equations (FPDEs), because each definition has its own limitations and benefits. Frenandez et al. [42] represented the series analysis of fractional operators involving Mittag-Leffler functions. Guariglia et al. [43] explained fractional derivatives of Riemann-Zeta function. Srivastava et al. [44] described the utility of fractional-order system in the biological population model with carrying capacity. The similarity analysis for strong shocks in non-ideal gas and numerical simulation of Ito coupled system have been analyzed by Arora et al. [11, 35]. Authors [6-8, 17, 18, 28, 30] suggested semi analytic and computational methodologies for obtaining the series and exact solutions of FODEs and FPDEs involving Laplace transform method, Sine-Cosine technique, homotopy analysis, variation iteration method, homotopy perturbation method, reduced differential transform and power series technique but these methods are useful to provide numerical or approximate series solutions. In our recent work, we are interested in application of Lie symmetry analysis [2], which has capability to provide us explicit and exact solution of fractional and integer-order problems. Authors [19-23, 25, 26] have imposed the invariance analysis and E-K operators on system of FPDEs with R-Lfractional derivative approach. Nowadays, researchers are working on conservation laws analysis of the systems. Sneddon [24] explained the application of fractional E-K differ-integral operators in beautiful manner, which is capable to convert the system of time fractional FPDEs into FODEs. Noether's theorem [27, 31, 33] established a relation between conservation laws and symmetry of differential equations and applied on FPDEs without Lagrangian operators. Recently, authors [34, 36-39, 47] provided invariance structure, explicit exactsolutions with power series solution and conservation analysis of Boussinesq-Burger’s system, Drinfeld-Sokolov-Satsuma-Hirota coupled KdV and m-KdV equations via Lie symmetry analysis. Biswas et al. [12-14] have worked on dual dispersion, power laws, conservation laws and optimal quasi-solitons by Lie symmetry analysis. Chauhan and Arora [29] has obtained the complete analysis of time fractional Kupershmidt equation. Recently, Gandhi et al. [40, 41, 50] have applied symmetry reduction on multi-ordered time-fractional KdV equations and Hirota-Satsoma-coupled Korteveg-de-Vries equations to obtain the explicit solutions with convergence and conservation laws; he concluded that the fractional-order parameter θ can control the output of solution of fractional mathematical models and in physical and mathematical aspects, the conservation laws play very crucial role to discuss the consistency of system. Zhang et al. [45] promoted the conservation laws of Fokkar-Plank equation with power diffusion. Bruzon et al. [46] focused on the study of similarity solutions and application of new conservation theorem on Cooper-Shepard-Sodano equation. New soliton solution of time-fractional Drinfeld-Sokolov-Satsuma-Hirota system in dispersive water waves has been illustrated by Ray et al. [48], he claimed the analytical solution with Adomial polynomials and Tanh-method. A generalized two-component Hunter-Saxton system has been studied by Yang et al. [49]. Dubey et al. [51] has suggested the copulation of fractional homotopy perturbation and fractional homotopy analysis method with Sumudu transforms on local-fractional Laplace equation. The Hydon method has been used by Chatibi et al. [52] to construct discrete symmetries for a family of ordinary, partial, and fractional differential equations. Using the Lie point symmetry, the generalized invariant structure of the (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation has been proposed by Chauhan et al. [53].

In present article, a nonlinear time fractional complex Hirota system of partial differential equations in Riemann-Liouville (RL) definition is considered by Lie symmetry approach. The main goal of this work is to elaborate the utilization of symmetry analysis on nonlinear fractional system of four equations with their conservation analysis. Hirota equations have many applications in propagation of sound and optical pulses in water crystals waveguide along with single mode fibers study. The complex Hirota system [21] is explained as follows:

$$\begin{cases} \dfrac{\partial^\theta z_1}{\partial t^\theta} + \dfrac{\partial^3 z_1}{\partial x^3} + 6\left(|\, z_1\,|^2 + |\, z_2\,|^2\right)\dfrac{\partial z_1}{\partial x} = 0, \\ \dfrac{\partial^\theta z_2}{\partial t^\theta} + \dfrac{\partial^3 z_2}{\partial x^3} + 6\left(|\, z_1\,|^2 + |\, z_2\,|^2\right)\dfrac{\partial z_2}{\partial x} = 0, \end{cases} \tag{1}$$

where $z_1(x,t) = u(x,t) + i\,v(x,t)$ *and* $z_2(x,t) = w(x,t) + i\,z(x,t),$

and $\left(|\, z_1\,|^2 + |\, z_2\,|^2\right) = u^2 + v^2 + w^2 + z^2 = \sum u^2.$

The system reduced to:

$$\begin{cases} \partial_t^\theta u + u_{xxx} + 6u_x\left(\sum u^2\right) = 0, \\ \partial_t^\theta v + v_{xxx} + 6v_x\left(\sum u^2\right) = 0, \\ \partial_t^\theta w + w_{xxx} + 6w_x\left(\sum u^2\right) = 0, \\ \partial_t^\theta z + z_{xxx} + 6z_x\left(\sum u^2\right) = 0, \end{cases} \tag{2}$$

here $\partial_t^\theta u, \partial_t^\theta v, \partial_t^\theta w$ and $\partial_t^\theta z$ are partial derivatives of *u*, *v*, *w* and *z* with fractional parameter $\theta\,(0 < \theta < 1)$ and independent variables '*x*' and '*t*' respectively. In continuation, section 2 is dedicated to preliminaries and invariance study of the system with the systematic use of Lie symmetry approach to find explicit solution in section 3. Section 4 is devoted to application of Erdyli-Kober operators for conversion of system of FPDEs into FODEs. In section 5, power series solution processed with convergence analysis under implicit function theorem. Finally, adjoint system and conservation laws are studied via Noether's theorem is proposed in section 6 and section 7. The conclusion and remarks are presented with references at the end of work.

2. Preliminaries

In this part, author would like to design some fundamentals and definitions for the sake of understanding the methodologies and key points, concerned with local fractional real order derivatives and integrals and their applications in fractional calculus.

Definition 2.1. The Caputo explained the fractional-order derivative of function $F(t)$ as

$$D_t^\theta(F(t)) = \frac{1}{\Gamma(\lambda-\theta)}\int_0^t (t-\rho)^{\lambda-\theta-1}F^\lambda(\rho)d\rho \;\; for \;\; \lambda-1 < \theta \le \lambda;\; \lambda \in N \;;\; t > 0. \tag{3}$$

Definition 2.2. The RL derived the definition of fractional-order derivative of F(t) as

$$D_t^\theta(F(t)) = \frac{1}{\Gamma(\lambda-\theta)}\frac{d^\lambda}{dt^\lambda}\int_0^t (t-\rho)^{\lambda-\theta-1}F(\rho)d\rho \;\; for \;\; \lambda-1 < \theta \le \lambda;\; \lambda \in N;\; t > 0. \tag{4}$$

Definition 2.3. Let the function $u(x,t)$ with variables '*x*' and *t > 0* then RL fractional partial order derivative is proposed as

$$\partial_t^{\theta}(u(x,t))=\begin{cases}\dfrac{1}{\Gamma(\lambda-\mu)}\dfrac{\partial^{\lambda}}{\partial t^{\lambda}}\int\limits_0^t (t-\rho)^{\lambda-\theta-1}u(\rho,x)d\rho \;\; for\;\; \lambda-1<\theta<\lambda,\;\; \lambda\in N,\\ \dfrac{\partial^{\lambda}u}{\partial t^{\lambda}} \quad for\;\; \theta=\lambda.\end{cases} \tag{5}$$

Definition 2.4. The Leibnitz described the product rule under application of RL fractional-order derivatives in the form

$$D_t^{\theta}(U.V)=\sum_{\lambda=0}^{\infty}\binom{\theta}{\lambda}D_t^{\theta-\lambda}(U).D_t^{\lambda}(V) \;\; ;\theta>0 \;\; with\;\; \binom{\theta}{\lambda}=\frac{(-1)^{\lambda}\theta\Gamma(n-\theta)}{\Gamma(1-\theta)\Gamma(\theta+1)}. \tag{6}$$

Definition 2.5. The E-Kober generalized fractional differential operator $\left(E_{\partial}^{\tau,\mu}\omega\right)(\zeta)$ is given by

$$\left(E_{\partial}^{\tau,\mu}\omega\right)(\zeta)=\prod_{\lambda=0}^{m-1}\left(\tau+\lambda-\frac{1}{\partial}z\frac{d}{dz}\right)(K_{\partial}^{\tau+\mu,m-\mu}\omega)(z) \;\; with\;\; \zeta>0,\; \partial>0 \;\; and\;\; \mu>0;$$
$$m=\begin{cases}[\mu]+1, & \mu\notin N\\ \mu\,, & \mu\in N\end{cases} \tag{7}$$

Definition 2.6. E-Kober generalized fractional-order integral operator $(K_{\partial}^{\tau,\mu}\omega)(\zeta)$ is

$$(K_{\partial}^{\tau,\mu}\omega)(\zeta)=\begin{cases}\dfrac{1}{\Gamma(\mu)}\int\limits_1^{\infty}(\upsilon-1)^{\mu-1}\upsilon^{-(\tau+\mu)}\omega(\zeta\upsilon^{1/\partial})d\upsilon, & \mu>0,\\ \omega(\zeta) \qquad\qquad\qquad\qquad , & \mu=0.\end{cases} \tag{8}$$

3. Methodology

In this section, authors would like to pursue the steps and process of fractional Lie symmetry reduction to coupled time fractional order system of FPDEs.

Let us assume the system of FPDEs with fractional order θ

$$\begin{cases}\partial_t^\theta u = F_1(t,x,u,v,w,u_x,v_x,w_x,z_x,u_{xx},v_{xx},w_{xx},z_{xx}...);\\ \partial_t^\theta v = F_2(t,x,u,v,w,u_x,v_x,w_x,z_x,u_{xx},v_{xx},w_{xx},z_{xx}...);\\ \partial_t^\theta w = F_3(t,x,u,v,w,u_x,v_x,w_x,z_x,u_{xx},v_{xx},w_{xx},z_{xx}...);\\ \partial_t^\theta z = F_4(t,x,u,v,w,u_x,v_x,w_x,z_x,u_{xx},v_{xx},w_{xx},z_{xx}...);\ 0<\theta<1.\end{cases} \tag{9}$$

The infinitesimal transformations with single parametric notation in fractional Lie symmetry analysis is expressed as:

$$\begin{cases}\bar{t} = \bar{t}(x,t,u,v,w,z;\varepsilon) = t+\varepsilon\tau(x,t,u,v,w,z)+o(\varepsilon^2);\\ \bar{x} = \bar{x}(x,t,u,v,w,z;\varepsilon) = x+\varepsilon\xi(x,t,u,v,w,z)+o(\varepsilon^2);\\ \bar{u} = \bar{u}(x,t,u,v,w,z;\varepsilon) = u+\varepsilon\eta(x,t,u,v,w,z)+o(\varepsilon^2);\\ \bar{v} = \bar{v}(x,t,u,v,w,z;\varepsilon) = v+\varepsilon\phi(x,t,u,v,w,z)+o(\varepsilon^2);\\ \bar{w} = \bar{w}(x,t,u,v,w,z;\varepsilon) = w+\varepsilon\mu(x,t,u,v,w,z)+o(\varepsilon^2);\\ \bar{z} = \bar{z}(x,t,u,v,w,z;\varepsilon) = z+\varepsilon\mu(x,t,u,v,w,z)+o(\varepsilon^2).\end{cases} \tag{10}$$

The vector field generated by infinitesimals is taken as:

$$X = \tau\,\partial_t + \xi\,\partial_x + \eta_1\,\partial_u + \eta_2\,\partial_v + \eta_3\,\partial_w + \eta_4\,\partial_z \tag{11}$$

with $\tau = \left.\frac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0}$, $\xi = \left.\frac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0}$, $\eta_1 = \left.\frac{d\bar{u}}{d\varepsilon}\right|_{\varepsilon=0}$, $\eta_2 = \left.\frac{d\bar{v}}{d\varepsilon}\right|_{\varepsilon=0}$, $\eta_3 = \left.\frac{dw}{d\varepsilon}\right|_{\varepsilon=0}$, $\eta_4 = \left.\frac{dz}{d\varepsilon}\right|_{\varepsilon=0}$.

Here, ξ, τ, η_1,η_2,η_3 and η_4 are obtained infinitesimals operators from (11), $\eta_1^{\theta,t},\eta_2^{\theta,t},\eta_3^{\theta,t}$ and $\eta_4^{\theta,t}$ are the fractional extended infinitesimals of order θ and $\eta_1^x,\eta_1^{xx},\eta_1^{xxx},\eta_2^x,\eta_2^{xx},\eta_2^{xxx},\eta_3^x,\eta_3^{xx},\eta_3^{xxx}$ and $\eta_4^x,\eta_4^{xx},\eta_4^{xxx}$ are extended infinitesimals of integer-order described

$$
\begin{aligned}
&\eta_1^{x} = D_x(\eta_1) - u_x D_x(\xi) - u_t D_x(\tau) \; ; \\
&\eta_1^{xx} = D_x(\eta_1^{x}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau) \; ; \\
&\eta_1^{xxx} = D_x(\eta_1^{xx}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau) \\
&\eta_2^{x} = D_x(\eta_2) - v_x D_x(\xi) - v_t D_x(\tau) \; ; \\
&\eta_2^{xx} = D_x(\eta_2^{x}) - v_{xx} D_x(\xi) - v_{xt} D_x(\tau); \\
&\eta_2^{xxx} = D_x(\eta_2^{xx}) - v_{xxx} D_x(\xi) - v_{xxt} D_x(\tau) \\
&\eta_3^{x} = D_x(\eta_3) - w_x D_x(\xi) - w_t D_x(\tau) \; ; \\
&\eta_3^{xx} = D_x(\eta_3^{x}) - w_{xx} D_x(\xi) - w_{xt} D_x(\tau) \; ; \\
&\eta_3^{xxx} = D_x(\eta_3^{xx}) - w_{xxx} D_x(\xi) - w_{xxt} D_x(\tau) \\
&\eta_4^{x} = D_x(\eta_4) - z_x D_x(\xi) - z_t D_x(\tau) \; ; \\
&\eta_4^{xx} = D_x(\eta_4^{x}) - z_{xx} D_x(\xi) - z_{xt} D_x(\tau); \\
&\eta_4^{xxx} = D_x(\eta_4^{xx}) - z_{xxx} D_x(\xi) - z_{xxt} D_x(\tau),
\end{aligned} \tag{12}
$$

where *'D_x'* is total derivative operator defined as:

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_{xx}} + \ldots + v_x \frac{\partial}{\partial v_x} + v_{xx} \frac{\partial}{\partial v_{xx}} + \ldots.. + w_x \frac{\partial}{\partial w_x} + w_{xx} \frac{\partial}{\partial w_{xx}} + \ldots.. \tag{13}$$

The extended infinitesimal function of θ-th order $\eta_1^{\theta,t}$ concerned to RL fractional derivative is described by:

$$\eta_1^{\theta,t} = D_t^{\theta}(\eta_1) + \xi.D_t^{\theta}(u_x) - D_t^{\theta}(\xi u_x) + D_t^{\theta}(D_t(\tau)u) - D_t^{\theta+1}(\tau u) + \tau.D_t^{\theta+1}(u). \tag{14}$$

Also, $D_t^{\theta+1}(f(t)) = D_t^{\theta}(D_t(f(t)),$ then above expression simplified to

$$\eta_1^{\theta,t} = D_t^{\theta}(\eta_1) + \xi.D_t^{\theta}(u_x) - D_t^{\theta}(\xi u_x) + \tau.D_t^{\theta}(u) - D_t^{\theta}(\tau u_t). \tag{15}$$

Applying the generalized Leibnitz rule on (15), we obtain

$$\eta_1^{\theta,t} = D_t^{\theta}(\eta_1) - \theta.D_t^{\theta}(\tau)\frac{\partial^{\theta} u}{\partial t^{\theta}} - \sum_{\lambda=1}^{\infty}\binom{\theta}{n} D_t^{\lambda}(\xi) D_t^{\theta-\lambda}(u_x) - \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda+1} D_t^{\lambda+1}(\tau) D_t^{\theta-\lambda}(u). \tag{16}$$

Using the generalized Leibnitz rule and chain rule (5) the term $D_t^{\theta}(\eta)$ in (16) can be defined as

$$D_t^\theta(\eta_1)=\frac{\partial^\theta\eta_1}{\partial t^\theta}+\left(\eta_{1u}\frac{\partial^\theta u}{\partial t^\theta}-u\frac{\partial^\theta(\eta_{1u})}{\partial t^\theta}\right)+\left(\eta_{1v}\frac{\partial^\theta v}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1v})}{\partial t^\theta}\right)+\left(\eta_{1w}\frac{\partial^\theta w}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1w})}{\partial t^\theta}\right)$$
$$\left(\eta_{1z}\frac{\partial^\theta z}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1z})}{\partial t^\theta}\right)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^n(\eta_{1u})}{\partial t^\lambda}D_t^{\theta-\lambda}(u)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1v})}{\partial t^\lambda}D_t^{\theta-\lambda}(v)+ \tag{17}$$
$$\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1w})}{\partial t^\lambda}D_t^{\theta-\lambda}(w)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1z})}{\partial t^\lambda}D_t^{\theta-\lambda}(z)+\sigma_1+\sigma_2+\sigma_3+\sigma_4,$$

where,

$$\begin{cases}\sigma_1=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{\lambda-m+k}\eta_1}{\partial t^{\lambda-m}\partial u^k},\\ \sigma_2=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{\lambda-m+k}\eta_1}{\partial t^{\lambda-m}\partial v^k},\\ \sigma_3=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-w)^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{\lambda-m+k}\eta_1}{\partial t^{\lambda-m}\partial w^k},\\ \sigma_4=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-z)^r\frac{\partial^m}{\partial t^m}(z^{k-r})\frac{\partial^{\lambda-m+k}\eta_1}{\partial t^{\lambda-m}\partial z^k}.\end{cases} \tag{18}$$

Finally, the expression for θ-th order extended infinitesimal $\eta_1^{\theta,t}$ of the form

$$\eta_1^{\theta,t}=\frac{\partial^\theta\eta_1}{\partial t^\theta}+\left(\eta_{1u}-\theta D_t(\tau)\right)\frac{\partial^\theta u}{\partial t^\theta}-u\frac{\partial^\theta(\eta_{1u})}{\partial t^\theta}+\left(\eta_{1v}\frac{\partial^\theta v}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1v})}{\partial t^\theta}\right)+$$
$$\left(\eta_{1w}\frac{\partial^\theta w}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1w})}{\partial t^\theta}\right)+\left(\eta_{1z}\frac{\partial^\theta z}{\partial t^\theta}-v\frac{\partial^\theta(\eta_{1z})}{\partial t^\theta}\right)+$$
$$\sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^n\eta_{1u}}{\partial t^n}-\binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(u)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1v})}{\partial t^\lambda}D_t^{\theta-\lambda}(v) \tag{19}$$
$$+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1w})}{\partial t^\lambda}D_t^{\theta-\lambda}(w)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{1z})}{\partial t^\lambda}D_t^{\theta-\lambda}(z)$$
$$-\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(u_x)+\sigma_1+\sigma_2+\sigma_3+\sigma_4.$$

Similarly, expressions for $\eta_2^{\theta,t}$ $\eta_3^{\theta,t}$ and $\eta_4^{\theta,t}$ are also obtained

$$\eta_2{}^{\theta,t} = \frac{\partial^\theta \eta_2}{\partial t^\theta} + \left(\eta_{2v} - \theta D_t(\tau)\right)\frac{\partial^\theta v}{\partial t^\theta} - v\frac{\partial^\theta(\eta_{2v})}{\partial t^\theta} + \left(\eta_{2u}\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{2u})}{\partial t^\theta}\right) +$$

$$\left(\eta_{2w}\frac{\partial^\theta w}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{2w})}{\partial t^\theta}\right) + \left(\eta_{2z}\frac{\partial^\theta z}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{2z})}{\partial t^\theta}\right) +$$

$$\sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^n \eta_{2v}}{\partial t^n} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(v) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{2u})}{\partial t^\lambda}D_t^{\theta-\lambda}(v) + \tag{20}$$

$$\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{2w})}{\partial t^\lambda}D_t^{\theta-\lambda}(w) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{2z})}{\partial t^\lambda}D_t^{\theta-\lambda}(z) -$$

$$\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(v_x) + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8.$$

$$\eta_3{}^{\theta,t} = \frac{\partial^\theta \eta_3}{\partial t^\theta} + \left(\eta_{3w} - \theta D_t(\tau)\right)\frac{\partial^\theta w}{\partial t^\theta} - w\frac{\partial^\theta(\eta_{3w})}{\partial t^\theta} + \left(\eta_{3u}\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{3u})}{\partial t^\theta}\right) +$$

$$\left(\eta_{3v}\frac{\partial^\theta v}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{3v})}{\partial t^\theta}\right) + \left(\eta_{3z}\frac{\partial^\theta z}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{3z})}{\partial t^\theta}\right) +$$

$$\sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^n \eta_{3w}}{\partial t^n} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(w) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{3u})}{\partial t^\lambda}D_t^{\theta-\lambda}(w) \tag{21}$$

$$+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{3v})}{\partial t^\lambda}D_t^{\theta-\lambda}(v) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{3z})}{\partial t^\lambda}D_t^{\theta-\lambda}(z) -$$

$$\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(w_x) + \sigma_9 + \sigma_{10} + \sigma_{11} + \sigma_{12}.$$

and

$$\eta_4{}^{\theta,t} = \frac{\partial^\theta \eta_4}{\partial t^\theta} + \left(\eta_{4z} - \theta D_t(\tau)\right)\frac{\partial^\theta z}{\partial t^\theta} - z\frac{\partial^\theta(\eta_{4z})}{\partial t^\theta} + \left(\eta_{4u}\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{4u})}{\partial t^\theta}\right)$$

$$+\left(\eta_{4w}\frac{\partial^\theta w}{\partial t^\theta} - u\frac{\partial^\theta(\eta_{4w})}{\partial t^\theta}\right) + \left(\eta_{4v}\frac{\partial^\theta z}{\partial t^\theta} - v\frac{\partial^\theta(\eta_{4z})}{\partial t^\theta}\right)$$

$$+\sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^n \eta_{4z}}{\partial t^n} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(z) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{4u})}{\partial t^\lambda}D_t^{\theta-\lambda}(u) \tag{22}$$

$$+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{4w})}{\partial t^\lambda}D_t^{\theta-\lambda}(w) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_{4v})}{\partial t^\lambda}D_t^{\theta-\lambda}(v)$$

$$-\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(z_x) + \sigma_{13} + \sigma_{14} + \sigma_{15} + \sigma_{16}.$$

$$\sigma_i = \sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-u)^r\,\frac{\partial^m}{\partial t^m}(u^{k-r})\,\frac{\partial^{\lambda-m+k}\eta}{\partial t^{\lambda-m}\partial u^k}. \qquad (23)$$

For expressions σ_i; i=5, 6, 7…16 given by (23) vanishes.

4. Lie symmetry reduction of nonlinear system of Hirota equations

Applying prolongation on set of Hirota system of FPDEs (2), we obtain

$$\begin{cases} \eta_1^{\theta,t} + \eta_1^{xxx} + 6\eta_1^x\left(\sum u^2\right) + 12u_x\left(u\eta_1 + v\eta_2 + w\eta_3 + z\eta_4\right) = 0; \\ \eta_2^{\theta,t} + \eta_2^{xxx} + 6\eta_2^x\left(\sum u^2\right) + 12v_x\left(u\eta_1 + v\eta_2 + w\eta_3 + z\eta_4\right) = 0; \\ \eta_3^{\theta,t} + \eta_3^{xxx} + 6\eta_3^x\left(\sum u^2\right) + 12w_x\left(u\eta_1 + v\eta_2 + w\eta_3 + z\eta_4\right) = 0; \\ \eta_4^{\theta,t} + \eta_4^{xxx} + 6\eta_4^x\left(\sum u^2\right) + 12z_x\left(u\eta_1 + v\eta_2 + w\eta_3 + z\eta_4\right) = 0. \end{cases} \qquad (24)$$

Using above explained facts in the methodology, substituting the values of extended infinitesimals and solving system of obtained fractional PDEs, we obtain the following infinitesimals

$$\begin{aligned} &\xi = \frac{p_1 x}{3} + p_2,\ \tau = \frac{p_1 t}{\theta},\ \eta_1 = \frac{-p_1 u}{3} + p_3 v + p_4 w + p_5 z, \\ &\eta_2 = -p_3 u - \frac{p_1 v}{3} + p_6 w + p_7 z, \\ &\eta_3 = -p_4 u - p_6 v - \frac{p_1 w}{3} + p_8 z, \\ &\eta_4 = -p_5 u - p_7 v - p_8 w - \frac{p_1 z}{3}, \end{aligned} \qquad (25)$$

where $p_i\,(i = 1, 2, \ldots 8)$ are components of standard basis of vector field

$$X_i = \xi\partial_x + \tau\partial_t + \eta_1\partial_u + \eta_2\partial_v + \eta_3\partial_w + \eta_4\partial_z. \qquad (26)$$

Lie algebra has the following vectors

$$
\begin{aligned}
&X_1 = \frac{x}{3}\partial_x + \frac{t}{\theta}\partial_t - \frac{u}{3}\partial_u - \frac{v}{3}\partial_v - \frac{w}{3}\partial_w - \frac{z}{3}\partial_z, \\
&X_2 = \partial_x,\ X_3 = v\partial_u - u\partial_v,\ X_4 = w\partial_u - u\partial_w, \\
&X_5 = z\partial_u - u\partial_z,\ \ X_6 = w\partial_v - v\partial_w, \\
&X_7 = z\partial_v - v\partial_z,\ X_8 = z\partial_w - w\partial_z.
\end{aligned} \tag{27}
$$

We are interested to evaluate the optimal solution of system of equations by choosing first infinitesimal generator with the help of characteristic equation.

$$
\frac{3dx}{x} = \frac{\theta dt}{t} = \frac{-3du}{u} = \frac{-3dv}{v} = \frac{-3dw}{w} = \frac{-3dz}{z}. \tag{28}
$$

Explicit solution obtained with 'α' as similarity variable

$$
u = t^{-\theta/3} f(\alpha),\ v = t^{-\theta/3} g(\alpha),\ w = t^{-\theta/3} h(\alpha),\ z = t^{-\theta/3} k(\alpha) \text{ and } \alpha = xt^{-\theta/3}. \tag{29}
$$

5. Application of EK differ-integral operators

Applying the R-L fractional derivative as:

$$
\begin{aligned}
\partial_t^{\theta} u &= \partial_t^{\lambda}\left[\frac{1}{\Gamma(\lambda-\theta)}\int_0^t (t-s)^{\lambda-\theta-1} s^{-\theta/3} f(xs^{-\theta/3})ds\right];\ q = t/s \\
&= \partial_t^{\lambda}\left[\frac{t^{\lambda-\frac{2\theta}{3}}}{\Gamma(\lambda-\theta)}\int_1^{\infty} (q-1)^{\lambda-\theta-1} q^{-\left(\lambda-\frac{2\theta}{3}+1\right)} f(\alpha q^{\theta/3})dq\right].
\end{aligned} \tag{30}
$$

Applying E-K integral operator then above expression found to be

$$
\partial_t^{\theta} u = \partial_t^{\lambda}\left[t^{\lambda-\frac{2\theta}{3}}\left(\kappa_{\frac{3}{\theta}}^{1-\frac{\theta}{3},\lambda-\theta} f\right)\alpha\right]. \tag{31}
$$

Also, using $t\,\partial_t f(\alpha) = tx\left(-\frac{\theta}{3}\right)t^{-\frac{\theta}{3}-1} f'(\alpha) = -\frac{\theta}{3}\alpha\frac{d}{d\alpha}(f(\alpha))$ in above equation (31), we obtain

$$\begin{aligned}\partial_t^{\theta} u &= \partial_t^{\lambda}\left(\partial_t\left[t^{\lambda-\frac{2\theta}{3}}\left(\kappa_{\frac{3}{\theta}}^{1-\frac{\theta}{3},\lambda-\theta} f\right)\alpha\right]\right) \\ &= \partial_t^{\lambda-1}\left[t^{\lambda-\frac{2\theta}{3}-1}\left(\lambda-\frac{4\theta}{3}-\frac{\theta}{3}\alpha\frac{d}{d\alpha}\right)\left(\kappa_{\frac{3}{\theta}}^{1-\frac{\theta}{3},\lambda-\theta} f\right)\alpha\right].\end{aligned} \tag{32}$$

Repeating above arguments (λ-1) times and applying E-K differential operator, we obtain

$$\partial_t^{\theta} u = t^{-\frac{2\theta}{3}}\left(P_{\frac{3}{\theta}}^{1-\frac{4\theta}{3},\theta} f\right)\alpha. \tag{33}$$

System of FPDEs is reduced to system of FODEs which is given by

$$\begin{cases}\left(P_{\frac{3}{\theta}}^{1-\frac{4\theta}{3},\theta} f\right)\alpha + f'''(\alpha) + 6f'(\alpha)\left(\sum f^2\right) = 0; \\ \left(P_{\frac{3}{\theta}}^{1-\frac{4\theta}{3},\theta} g\right)\alpha + g'''(\alpha) + 6g'(\alpha)\left(\sum f^2\right) = 0; \\ \left(P_{\frac{3}{\theta}}^{1-\frac{4\theta}{3},\theta} h\right)\alpha + h'''(\alpha) + 6h'(\alpha)\left(\sum f^2\right) = 0; \\ \left(P_{\frac{3}{\theta}}^{1-\frac{4\theta}{3},\theta} k\right)\alpha + k'''(\alpha) + 6k'(\alpha)\left(\sum f^2\right) = 0,\end{cases} \tag{34}$$

where $\sum f^2 = f^2 + g^2 + h^2 + k^2$.

6. Power series solution of system and its convergence:

Set $$f(\alpha) = \sum_{n=1}^{\infty} a_n \alpha^n,\ g(\alpha) = \sum_{n=1}^{\infty} b_n \alpha^n,\ h(\alpha) = \sum_{n=1}^{\infty} c_n \alpha^n,\ f(\alpha) = \sum_{n=1}^{\infty} d_n \alpha^n. \tag{35}$$

From above equation, we obtained

$$\begin{cases} \sum_{n=0}^{\infty} \dfrac{\Gamma\left(2-\frac{4\theta}{3}+\frac{n\theta}{3}\right)}{\Gamma\left(2-\frac{\theta}{3}+\frac{n\theta}{3}\right)} a_n \alpha^n + \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} \alpha^n + 6\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} \alpha^n\right)[S] = 0; \\ \sum_{n=0}^{\infty} \dfrac{\Gamma\left(2-\frac{4\theta}{3}+\frac{n\theta}{3}\right)}{\Gamma\left(2-\frac{\theta}{3}+\frac{n\theta}{3}\right)} b_n \alpha^n + \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) b_{n+3} \alpha^n + 6\left(\sum_{n=0}^{\infty} (n+1) b_{n+1} \alpha^n\right)[S] = 0; \\ \sum_{n=0}^{\infty} \dfrac{\Gamma\left(2-\frac{4\theta}{3}+\frac{n\theta}{3}\right)}{\Gamma\left(2-\frac{\theta}{3}+\frac{n\theta}{3}\right)} c_n \alpha^n + \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) c_{n+3} \alpha^n + 6\left(\sum_{n=0}^{\infty} (n+1) c_{n+1} \alpha^n\right)[S] = 0; \\ \sum_{n=0}^{\infty} \dfrac{\Gamma\left(2-\frac{4\theta}{3}+\frac{n\theta}{3}\right)}{\Gamma\left(2-\frac{\theta}{3}+\frac{n\theta}{3}\right)} d_n \alpha^n + \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) d_{n+3} \alpha^n + 6\left(\sum_{n=0}^{\infty} (n+1) d_{n+1} \alpha^n\right)[S] = 0, \end{cases} \tag{36}$$

$$\text{where } S = \left(\sum_{n=1}^{\infty} a_n \alpha^n\right)^2 + \left(\sum_{n=1}^{\infty} b_n \alpha^n\right)^2 + \left(\sum_{n=1}^{\infty} c_n \alpha^n\right)^2 + \left(\sum_{n=1}^{\infty} d_n \alpha^n\right)^2. \tag{37}$$

Substituting n=0 in (36), we get

$$\begin{cases} a_3 = -\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)} a_0 - a_1 S_0; \\ b_3 = -\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)} b_0 - b_1 S_0; \\ c_3 = -\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)} c_0 - c_1 S_0; \\ d_3 = -\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)} d_0 - d_1 S_0, \end{cases} \quad \text{where } S_0 = {a_0}^2 + {b_0}^2 + {c_0}^2 + {d_0}^2, \tag{38}$$

and

$$\begin{cases} a_{n+3} = \dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}a_n - \dfrac{6}{(n+3)(n+2)}\sum_{k=0}^{n} a_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right); \\ b_{n+3} = \dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}b_n - \dfrac{6}{(n+3)(n+2)}\sum_{k=0}^{n} b_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right); \\ c_{n+3} = \dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}c_n - \dfrac{6}{(n+3)(n+2)}\sum_{k=0}^{n} c_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right); \\ d_{n+3} = \dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}d_n - \dfrac{6}{(n+3)(n+2)}\sum_{k=0}^{n} d_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right). \end{cases} \tag{39}$$

So the explicit power series solution of system is given by:

$$\begin{cases} f(\alpha) = a_0 + a_1\left(xt^{-\theta/3}\right) + a_2\left(xt^{-\theta/3}\right)^2 + \left[-\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)}a_0 - a_1S_0\right]\left(xt^{-\theta/3}\right)^3 + \sum_{n=1}^{\infty} a_{n+3}\left(xt^{-\theta/3}\right)^{n+3} \\ g(\alpha) = b_0 + b_1\left(xt^{-\theta/3}\right) + b_2\left(xt^{-\theta/3}\right)^2 + \left[-\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)}b_0 - b_1S_0\right]\left(xt^{-\theta/3}\right)^3 + \sum_{n=1}^{\infty} b_{n+3}\left(xt^{-\theta/3}\right)^{n+3} \\ h(\alpha) = c_0 + c_1\left(xt^{-\theta/3}\right) + c_2\left(xt^{-\theta/3}\right)^2 + \left[-\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)}c_0 - c_1S_0\right]\left(xt^{-\theta/3}\right)^3 + \sum_{n=1}^{\infty} c_{n+3}\left(xt^{-\theta/3}\right)^{n+3} \\ k(\alpha) = d_0 + d_1\left(xt^{-\theta/3}\right) + d_2\left(xt^{-\theta/3}\right)^2 + \left[-\dfrac{\Gamma(2-4\theta/3)}{3!\Gamma(2-\theta/3)}d_0 - d_1S_0\right]\left(xt^{-\theta/3}\right)^3 + \sum_{n=1}^{\infty} d_{n+3}\left(xt^{-\theta/3}\right)^{n+3} \end{cases} \tag{40}$$

$$
\begin{cases}
|a_{n+3}| = \left|\dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}\right||a_n| - \left|\dfrac{6}{(n+3)(n+2)}\right|\left|\sum_{k=0}^{n} a_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right)\right|; \\
|b_{n+3}| = \left|\dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}\right||b_n| - \left|\dfrac{6}{(n+3)(n+2)}\right|\left|\sum_{k=0}^{n} b_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right)\right|; \\
|c_{n+3}| = \left|\dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}\right||c_n| - \left|\dfrac{6}{(n+3)(n+2)}\right|\left|\sum_{k=0}^{n} c_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right)\right|; \\
|d_{n+3}| = \left|\dfrac{n!}{(n+3)!}\dfrac{\Gamma\left(2-\dfrac{4\theta}{3}+\dfrac{n\theta}{3}\right)}{\Gamma\left(2-\dfrac{\theta}{3}+\dfrac{n\theta}{3}\right)}\right||d_n| - \left|\dfrac{6}{(n+3)(n+2)}\right|\left|\sum_{k=0}^{n} d_{n+1-k}\left(a_{n-k}^2+b_{n-k}^2+c_{n-k}^2+d_{n-k}^2\right)\right|.
\end{cases} \tag{41}
$$

It is well known that for n < m, the expressions like $\left|\dfrac{\Gamma(n)}{\Gamma(m)}\right| < 1$ and 'M' is maximum of the arbitrary coefficients involved in the set of equations with introduction of majorant series. Now, we introduce some another power series

$$
U(\chi)=\sum_{n=0}^{\infty} p_n \chi^n ; V(\chi)=\sum_{n=0}^{\infty} q_n \chi^n ; W(\chi)=\sum_{n=0}^{\infty} r_n \chi^n \text{ and } Y(\chi)=\sum_{n=0}^{\infty} s_n \chi^n, \tag{42}
$$

where, $p_n=|a_n|$, $q_n=|b_n|$, $r_n=|c_n|$, $s_n=|d_n|$, $n = 1, 2, 3, \ldots$, then, we can have

$$
\begin{cases}
p_{n+3} = M\left[p_n + \sum_{k=0}^{n} p_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right]; \\
q_{n+3} = M\left[q_n + \sum_{k=0}^{n} q_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right]; \\
r_{n+3} = M\left[r_n + \sum_{k=0}^{n} r_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right]; \\
s_{n+3} = M\left[s_n + \sum_{k=0}^{n} s_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right].
\end{cases} \tag{43}
$$

Assuming the implicit functions of system with independent variable χ.

$$U_1(\chi,U)=U(\chi)-p_0-p_1\chi-p_2\chi^2-p_3\chi^3-M\left[p_n+\sum_{k=0}^{n}p_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right];$$
$$V_1(\chi,V)=V(\chi)-q_0-q_1\chi-q_2\chi^2-q_3\chi^3-M\left[q_n+\sum_{k=0}^{n}q_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right];$$
$$W_1(\chi,W)=W(\chi)-r_0-r_1\chi-r_2\chi^2-r_3\chi^3-M\left[r_n+\sum_{k=0}^{n}r_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right];\quad (44)$$
$$Y_1(\chi,Y)=Y(\chi)-s_0-s_1\chi-s_2\chi^2-s_3\chi^3-M\left[s_n+\sum_{k=0}^{n}s_{n+1-k}\left(p_{n-k}^2+q_{n-k}^2+r_{n-k}^2+s_{n-k}^2\right)\right].$$

Here, U_1, V_1, W_1 and Y_1 are analytic in a neighborhood of $(0, p_0)$, $(0, q_0)$, $(0, r_0)$ and $(0, s_0)$ respectively, where $U_1(0, p_0) = 0$; $V_1(0,q_0)= 0$, $W_1(0,r_0) = 0$ and $Y_1(0,s_0) = 0$ with $\frac{\partial}{\partial U}(U_1(0,p_0))\neq 0$ $\frac{\partial}{\partial V}(V_1(0,q_0))\neq 0$ $\frac{\partial}{\partial W}(W_1(0,r_0))\neq 0$ and $\frac{\partial}{\partial Y}(Y_1(0,s_0))\neq 0$ then, with Implicit function theorem [40, 50], we reached at convergence of power series solution.

7. Conservation laws

The study of classical and fractional PDEs is incomplete without the discussion of conservation laws for consistency and stability of system. We obtained the distinct conserved vectors for distinct infinitesimal generators associated with system of time fractional PDEs due to existence of one-to-one correspondence between them. These laws can be evaluated with the aid of Noether's theorem [31, 33]. For the sake of conservation laws, we should discuss the adjoint system along with Lagrangian and components of conserved vectors ω^t and ω^x , which has the continuity equation as follows:

$$D_t(\omega^t)+D_x(\omega^x)=0. \quad (45)$$

The Lagrangian of system of FPDEs with four new dependent variables P, Q, R and $S$of independent variables t and x is in the form

$$\begin{aligned}\ell = P\left(\partial_t^\theta u+\partial_x^3 u+6\partial_x u\left(\sum u^2\right)\right)+Q\left(\partial_t^\theta v+\partial_x^3 v+6\partial_x v\left(\sum u^2\right)\right)+\\ R\left(\partial_t^\theta w+\partial_x^3 w+6\partial_x w\left(\sum u^2\right)\right)+S\left(\partial_t^\theta z+\partial_x^3 z+6\partial_x z\left(\sum u^2\right)\right).\end{aligned} \quad (46)$$

With Euler-Lagranges (E-L) equations also known as adjoint equations described as:

$$\frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta v} = \frac{\delta \ell}{\delta w} = \frac{\delta \ell}{\delta z} = 0, \tag{47}$$

with E-L operators $\frac{\delta}{\delta u^i} = \frac{\partial}{\partial u^i} + (D_t^\theta)^* \frac{\partial}{\partial D_t^\theta u^i} - D_x\left(\frac{\partial}{\partial u^i{}_x}\right) + D_{xx}\left(\frac{\partial}{\partial u^i{}_{xx}}\right) - D_{xxx}\left(\frac{\partial}{\partial u^i{}_{xxx}}\right),$ (48)

and $(D_t^\theta)^*$ represents the adjoint operator to D_t^θ, which is defined in right-sided Caputo time-fractional derivative of order 'θ' as:

$$(D_t^\theta)^* = {}_t^C D_T^\theta u = \frac{(-1)^n}{\Gamma(n-\theta)} \int_t^T (\upsilon - t)^{n-1-\theta} D_\upsilon^\theta u(\upsilon, x) d\upsilon; \quad n = [\theta] + 1. \tag{49}$$

The adjoint system of equations formed by system of FPDEs

$$\begin{cases} (D_t^\theta)^* P - P_{xxx} + 6P_x\left(\sum u^2\right) - 12P\left(\sum uu_x\right) = 0; \\ (D_t^\theta)^* Q - Q_{xxx} + 6Q_x\left(\sum u^2\right) - 12Q\left(\sum uu_x\right) = 0; \\ (D_t^\theta)^* R - R_{xxx} + 6R_x\left(\sum u^2\right) - 12R\left(\sum uu_x\right) = 0; \\ (D_t^\theta)^* S - S_{xxx} + 6S_x\left(\sum u^2\right) - 12S\left(\sum uu_x\right) = 0, \end{cases}$$

$$\text{where } \sum u^2 = u^2 + v^2 + w^2 + z^2, \tag{50}$$

$$\text{and } \sum uu_x = uu_x + vv_x + ww_x + zz_x,$$

for two independent variables *x* and *t* with dependents *u*, *v*, *w* and *z* respectively, the components of conserved vectors ω^t and ω^x are expressed as

$$\begin{aligned} \omega^x &= \xi\ell + W_j\left[\frac{\partial \ell}{\partial u_x^j} - D_x\left(\frac{\partial \ell}{\partial u_{xx}^j}\right) + D_x^2\left(\frac{\partial \ell}{\partial u_{xxx}^j}\right)\right] + D_x(W_j)\left[\frac{\partial \ell}{\partial u_{xx}^j} - D_x\left(\frac{\partial \ell}{\partial u_{xxx}^j}\right)\right] + D_x^2(W_j)\left[\frac{\partial \ell}{\partial u_{xxx}^j}\right]; \\ \omega^t &= \tau\ell + D_t^{\theta-1}(W_j)\frac{\partial \ell}{\partial D_t^\theta u^j} + I\left(W_j,\ D_t \frac{\partial \ell}{\partial D_t^\theta u^j}\right), \end{aligned} \tag{51}$$

Here $W_j = \eta_j - \xi_j u_x - \tau_j u_t$, ℓ is defined above in (46) and 'I' is integral defined as

$$I(f, g) = \frac{1}{\Gamma(1-\theta)} \int_0^t \int_t^T \frac{f(s,x) g(\mu, x)}{(\mu - s)^\theta} d\mu ds. \tag{52}$$

Case 1: For X$_1$, Lie characteristic functions are $W_1=-\frac{u}{3}-\frac{x}{3}u_x-\frac{t}{\theta}u_t$,$W_2=-\frac{v}{3}-\frac{x}{3}v_x-\frac{t}{\theta}v_t$ $W_3=-\frac{w}{3}-\frac{x}{3}w_x-\frac{t}{\theta}w_t$,$W_4=-\frac{z}{3}-\frac{x}{3}z_x-\frac{t}{\theta}z_t$ and components of conserved vectors obtained as given below:

$$
\begin{aligned}
\omega^x &= \frac{x}{3}\ell+\left(-\frac{u}{3}-\frac{x}{3}u_x-\frac{t}{\theta}u_t\right)\left[6P\left(\sum u^2\right)+P_{xx}\right]+P_x\left(\frac{2u_x}{3}+\frac{xu_{xx}}{3}+\frac{tu_{tx}}{\theta}\right)\\
&-P\left(\frac{u_{xx}}{3}+\frac{xu_{xxx}}{3}-\frac{tu_{txx}}{\theta}\right)+\left(-\frac{v}{3}-\frac{x}{3}v_x-\frac{t}{\theta}v_t\right)\left[6Q\left(\sum u^2\right)+Q_{xx}\right]\\
&+Q_x\left(\frac{2v_x}{3}+\frac{xv_{xx}}{3}+\frac{tv_{tx}}{\theta}\right)-Q\left(\frac{v_{xx}}{3}+\frac{xv_{xxx}}{3}-\frac{tv_{txx}}{\theta}\right)+\left(-\frac{w}{3}-\frac{x}{3}w_x-\frac{t}{\theta}w_t\right)\left[6R\left(\sum u^2\right)+R_{xx}\right] \quad (53)\\
&+R_x\left(\frac{2w_x}{3}+\frac{xw_{xx}}{3}+\frac{tw_{tx}}{\theta}\right)-R\left(\frac{w_{xx}}{3}+\frac{xw_{xxx}}{3}-\frac{tw_{txx}}{\theta}\right)+\left(-\frac{z}{3}-\frac{x}{3}z_x-\frac{t}{\theta}z_t\right)\left[6S\left(\sum u^2\right)+S_{xx}\right]\\
&+S_x\left(\frac{2z_x}{3}+\frac{xz_{xx}}{3}+\frac{tz_{tx}}{\theta}\right)-S\left(\frac{z_{xx}}{3}+\frac{xz_{xxx}}{3}-\frac{tz_{txx}}{\theta}\right),
\end{aligned}
$$

$$
\begin{aligned}
\omega^t &= PD_t^{\theta-1}\left(-\frac{u}{3}-\frac{x}{3}u_x-\frac{t}{\theta}u_t\right)+I\left(-\frac{u}{3}-\frac{x}{3}u_x-\frac{t}{\theta}u_t,P_t\right)+QD_t^{\theta-1}\left(-\frac{v}{3}-\frac{x}{3}v_x-\frac{t}{\theta}v_t\right)+\\
&I\left(-\frac{v}{3}-\frac{x}{3}v_x-\frac{t}{\theta}v_t,Q_t\right)+RD_t^{\theta-1}\left(-\frac{w}{3}-\frac{x}{3}w_x-\frac{t}{\theta}w_t\right)+I\left(-\frac{w}{3}-\frac{x}{3}w_x-\frac{t}{\theta}w_t,R_t\right)+ \quad (54)\\
&SD_t^{\theta-1}\left(-\frac{z}{3}-\frac{x}{3}z_x-\frac{t}{\theta}z_t\right)+I\left(-\frac{z}{3}-\frac{x}{3}z_x-\frac{t}{\theta}z_t,S_t\right).
\end{aligned}
$$

Case 2: For X$_2$,$W_1=-u_x$, $W_2=-v_x$, $W_3=-w_x$, and $W_4=-z_x$, we have

$$
\begin{aligned}
\omega^x &= \ell-u_x\left[6P\sum u^2+P_{xx}\right]-u_{xx}(P_x)-u_{xxx}P-v_x\left[6Q\sum u^2+Q_{xx}\right]-v_{xx}(Q_x)-u_{xxx}Q\\
&-w_x\left[6R\sum u^2+R_{xx}\right]-w_{xx}(R_x)-w_{xxx}R-z_x\left[6S\sum u^2+S_{xx}\right]-z_{xx}(S_x)-z_{xxx}S,
\end{aligned}
\quad (55)
$$

$$
\begin{aligned}
\omega^t &= -PD_t^{\theta-1}(u_x)+I(-u_x,P_t)-QD_t^{\theta-1}(v_x)+I(-v_x,Q_t)\\
&-RD_t^{\theta-1}(w_x)+I(-w_x,R_t)-SD_t^{\theta-1}(z_x)+I(-z_x,S_t).
\end{aligned}
\quad (56)
$$

Case 3: For X$_3$, $W_1=v$, $W_2=-u$, $W_3=0$, and $W_4=0$, we have

$$
\begin{aligned}
\omega^x &= v\left[6P\left(\sum u^2\right)+P_{xx}\right]-v_xP_x+Pv_{xx}-u\left[6Q\left(\sum u^2\right)+Q_{xx}\right]+u_xQ_x+Qu_{xx},\\
\omega^t &= PD_t^{\theta-1}(v)+I(v,P_t)-QD_t^{\theta-1}(u)+I(-u,Q_t).
\end{aligned}
\quad (57)
$$

Case 4: For X4, $W_1 = w,\ W_2 = 0,\ W_3 = -u,$ and $W_4 = 0$, we have

$$\begin{aligned}\omega^x &= w\left[6P\left(\sum u^2\right) + P_{xx}\right] - w_x P_x + P w_{xx} - u\left[6R\left(\sum u^2\right) + R_{xx}\right] + u_x R_x + R u_{xx},\\ \omega^t &= P D_t^{\theta-1}(w) + I(w, P_t) - R D_t^{\theta-1}(u) + I(-u, R_t).\end{aligned} \tag{58}$$

Case 5: For X5, $W_1 = z,\ W_2 = 0,\ W_3 = 0,$ and $W_4 = -u$, we have

$$\begin{aligned}\omega^x &= z\left[6P\left(\sum u^2\right) + P_{xx}\right] - z_x P_x + P z_{xx} - u\left[6S\left(\sum u^2\right) + S_{xx}\right] + u_x S_x + S u_{xx},\\ \omega^t &= P D_t^{\theta-1}(z) + I(z, P_t) - S D_t^{\theta-1}(u) + I(-u, S_t).\end{aligned} \tag{59}$$

Case 6: For X6, $W_1 = 0,\ W_2 = w,\ W_3 = -v,$ and $W_4 = 0$, we have

$$\begin{aligned}\omega^x &= w\left[6Q\left(\sum u^2\right) + Q_{xx}\right] - w_x Q_x + Q w_{xx} - v\left[6R\left(\sum u^2\right) + R_{xx}\right] + v_x R_x + R v_{xx},\\ \omega^t &= Q D_t^{\theta-1}(w) + I(w, Q_t) - R D_t^{\theta-1}(v) + I(-v, R_t).\end{aligned} \tag{60}$$

Case 7: For X7, $W_1 = 0,\ W_2 = z,\ W_3 = 0,$ and $W_4 = -v$, we have

$$\begin{aligned}\omega^x &= z\left[6Q\left(\sum u^2\right) + Q_{xx}\right] - z_x Q_x + Q z_{xx} - v\left[6S\left(\sum u^2\right) + S_{xx}\right] + v_x S_x + S v_{xx},\\ \omega^t &= Q D_t^{\theta-1}(w) + I(w, Q_t) - S D_t^{\theta-1}(v) + I(-v, S_t).\end{aligned} \tag{61}$$

Case 8: For X8, $W_1 = 0,\ W_2 = 0,\ W_3 = z,$ and $W_4 = -w$, we have

$$\begin{aligned}\omega^x &= z\left[6R\left(\sum u^2\right) + R_{xx}\right] - w_x R_x + R w_{xx} - w\left[6S\left(\sum u^2\right) + S_{xx}\right] + w_x S_x + S w_{xx},\\ \omega^t &= R D_t^{\theta-1}(z) + I(z, R_t) - S D_t^{\theta-1}(w) + I(-w, S_t).\end{aligned} \tag{62}$$

8. Conclusions

The invariance analysis for fractional-order nonlinear coupled Hirota system of PDEs has been done successfully by the application of Lie symmetry reduction. The obtained infinitesimals and generators are used for reduction of the system into nonlinear FODEs via validity of E-K operators in R-L fractional derivative sense. The power series solution and its convergence are discussed by imposition of majorant series and Implicit function theorem. We applied Euler-Lagranges operators to create the adjoint system of the system and conservation analysis with the use of Noether’s theorem for stability and consistency of system. This work may be applicable to fractional fluid flow, propagation of sound, single mode fiber study and other nonlinear evolution problems.

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