General decay of a nonlinear viscoelastic wave equation with Balakrishn\^{a}n-Taylor damping and a delay involving variable source

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Abstract

This paper is devoted to the stability of a viscoelastic problem with Balakrishn\^{a}n-Taylor damping and time delay involving variable-exponent nonlinearity. Under some assumptions on the relaxation function, we establish the general decay estimate for the energy via suitable Lyapunov functionals. The problem considered is novel and meaningful because the presence of the flutter panel equation and the spillover problem with memory and variable exponents time delay control. Our result generalizes and improves previous conclusion in the literature.

GENERAL DECAY OF A NONLINEAR VISCOELASTIC WAVE EQUATION WITH BALAKRISHNÂN-TAYLOR DAMPING AND A DELAY INVOLVING VARIABLE SOURCE

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Abstract. This paper is devoted to the stability of the following viscoelastic problem with Balakrishnân-Taylor damping and time delay involving variable-exponent nonlinearity

$$u_{tt} - \mathcal{M}\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s + \mu_1 |u_t|^{p(\cdot)-2} u_t$$

$$+\mu_2 |u_t(t-\tau)|^{p(\cdot)-2} u_t(t-\tau) = 0 \quad \text{in } \Omega\times\mathbb{R}^+,$$

where Ω is a bounded domain of $\mathbb{R}^n$, $p(.) : \overline{\Omega} \to \mathbb{R}$ is a measurable function, $g > 0$ is a memory kernel that decay exponentially, $\alpha \geq 0$ is the potential, and

$$\mathcal{M}\left(\|\nabla u\|_2^2\right) = a + b\,\|\nabla u(t)\|_2^2 + \sigma\int_\Omega \nabla u \nabla u_t dx$$

for some constants $a > 0$, $b \geq 0$, $\sigma > 0$. Under some assumptions on the relaxation function, we establish the general decay estimate for the energy via suitable Lyapunov functionals.

The problem considered is novel and meaningful because the presence of the flutter panel equation and the spillover problem with memory and variable exponents time delay control. Our result generalizes and improves previous conclusion in the literature.

1. Introduction

In recent years, many authors have paid attention to the equations with variable exponents of nonlinearities. This is partially due to the large employment of these equations to model several physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through porous media, and image processing gives rise to equations

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with nonstandard growth conditions. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in the gradient of the unknown solution and with variable exponents of nonlinearity. The study of these systems is based on the use of the Lebesgue and Sobolev spaces with variable exponents (see for instance [15, 14, 40]). More details on these problems can be found in previous studies [30, 34, 32, 31, 33, 35, 36, 38, 37, 43, 44, 45] and references therein.

In this paper, we are concerned with the asymptotic behavior of weak solutions to a weakly damped viscoelastic wave equation with Balakrishnân–Taylor damping and delay term involving the variable-exponent nonlinearities

$$\begin{cases} u_{tt} - \mathcal{M}\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)\mathrm{d}s \\ +\mu_1 |u_t|^{p(x)-2} u_t + \mu_2 |u_t(t-\tau)|^{p(x)-2} u_t(t-\tau) = 0 \text{ in } \Omega\times(0,\infty) \\ u(x,t) = 0 \text{ on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \\ u_t(x,t) = j_0(x,t-\tau) \text{ in } \Omega\times(0,\tau), \end{cases} \tag{1.1}$$

where $\mathcal{M}\left(\|\nabla u\|_2^2\right) = a + b\|\nabla u(t)\|_2^2 + \sigma\int_\Omega \nabla u\nabla u_t dx$, $a>0$, $b\geq 0$, $\sigma>0$, $\mu_1\geq 0$, μ_2 is a real number, $\tau>0$ is the time delay, $g>0$ is a memory kernel, $\alpha\geq 0$ is the potential, Δ stands for the Laplacian with respect to the spatial variables. A great deal of attention has been given to model several phenomena such as vibrations of elastic strings and plates, as in the case where, $g=0$, $\mu_1=\mu_2=0$, equation (1.1)) is described by Kirchhoff's original equation

$$\rho h\frac{\partial^2 u}{\partial t^2} = \left\{p_0 + \frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f,\quad 0\leq x\leq L,\ t\geq 0, \tag{1.2}$$

which was first introduced in 1883 by Kirchhoff [29], to the study of the oscillations of stretched strings and plates and called the wave equation of Kirchhoff type, where $u=u(x,t)$ is the lateral deflection, E is Young's modulus, ρ is the mass density, h is the cross-section area, L is the length, p_0 is the initial axial tension, and f is the external force. There are large discussions concerning the Kirchhoff equation. In the sequel, we would like to mention some considerable efforts on this topic.

The local solutions in time, well-posedness, and solvability of the Kirchhoff type equation (1.2), has been well-studied in general dimensions and domains by various authors (see, for examples, [17, 16, 18, 19, 20, 21, 25, 23, 28] and the references therein).

In the presence of the Balakrishnân-Taylor damping term ($\sigma>0$), problem (1.1) in the case when $p>1$ is constant, is related to the flutter panel equation and the spillover problem with memory and time delay control. Balakrishnân and Taylor [5] and Bass and Zes [6] introduced Balakrishnân–Taylor damping which arises from a wind tunnel experiment at supersonic speeds, see [7, 23, 8, 24, 26].

Time delays arise in many physical, chemical, biological, thermal and economic phenomena because these phenomena depend not only on the present state but also on the past history of the system in a more complicated way (see, for example, [9, 11, 10])

Regarding the viscoelastic wave equation with delay several authors discussed on existence and stability of the solutions under appropriate conditions on μ_1,μ_2, and g (see e.g. [1]). For the related problems, we also refer [2, 12, 3, 4, 27, 13].

The terminology variable exponents comes from the fact that $p(.)$ is a function , not a real number. This term $\mu_1 |u_t|^{p(.)-2} u_t + \mu_2 |u_t(t-\tau)|^{p(.)-2} u_t(t-\tau)$ is a generalization of $\mu_1 u_t + \mu_2 u_t(t-\tau)$, which corresponds to $p(.) > 1$. As a matter of fact, (1.1) can be cast as an extension to the variable case of the second-order viscoelastic wave equation with variable growth conditions

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0 \text{ in } \Omega\times\mathbb{R}^+, \tag{1.3}$$

which is obtained when considering $\mu_1 |u_t|^{p(.)-2} u_t + \mu_2 |u_t(t-\tau)|^{p(.)-2} u_t(t-\tau)$. Equation (1.3) is a well-known appears in the treatment of fluid dynamics, a model for electrorheological fluids [41]. On the other hand, results for the viscoelastic wave equation with Balakrishnân–Taylor damping, delay term, and variable growth conditions are limited and rare, and the literature on these equations is much less extended see [22]. In particular, in [2], the authors considered this class of equations under some suitable assumptions, they showed that the general decay exponentially, see similar work in [27].

The purpose of this paper is to generalize previous some results. In particular, we will establish in this case with the relaxation function and specified initial data by using special Lypunouv functionals a general decay estimate for the energy, which depends on the behavior of the relation function, and which is not necessarily decaying in a polynomial or exponential shape.

This paper is composed of two sections in addition to the introduction. In Section 2, we recall the definitions of the variable-exponent Lebesgue and Sobolev spaces and present some of their relevant properties. We also state there our main results. In Section 3, we prove our result showing the general decay of a solution to (1.1) with a small initial value (u_0, u_1).

2. Functional setting and main results

In this section, we describe the functional setting in which we shall work and state our main results.

Let us start by introducing the Lebesgue and Sobolev spaces with variable exponent. Here we refer mainly to [14, 39, 42].

Throughout the rest of the paper, we assume that Ω is a bounded open domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ. Moreover, in what follows, if not stated differently, we will always assume that $p : \overline{\Omega} \to (1, +\infty)$ is a measurable function and we will denote

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega}[\,p(x)] \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega}[\,p(x)].$$

We then define the variable-exponent space $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \varrho_{p(.),\Omega}(v) := \int_\Omega |v(x)|^{p(x)}\,\mathrm{d}x < +\infty \right\},$$

which is a Banach space equipped with the Luxemburg norm

$$\|u\|_{p(.),\Omega} := \inf\left\{ \lambda > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

Variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects (see for instance [42]). In particular, it follows directly from the definition of the norm that

$$\min\left(\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right)\le \varrho_{p(.),\Omega}(u)\le \max\left(\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right). \tag{2.1}$$

In this section, we outline the variational framework for problem (1.1) and give some preliminary Lemmas.

Given a measurable function $p:\overline{\Omega}\to[p^-,p^+]\subset(2,\infty)$, $p^\pm=\text{const}$, we define the Lebesgue space with variable exponent

$$L^{p(.)}(\Omega)=\left\{\begin{array}{r} u:\Omega\to\mathbb{R}: u \text{ measurable functions on } \Omega,\\ \int\limits_\Omega |u(x)|^{p(x)}\,\mathrm{d}x<\infty.\end{array}\right\}$$

equipped with the Luxemburg norm,

$$\|u\|_{p(.)}=\inf\left\{\lambda>0,\ \int\limits_\Omega\left|\frac{u}{\lambda}\right|^{p(x)}\mathrm{d}x\le 1\right\}$$

is a Banach space. We also assume that p satisfies the following Zhikov–Fan uniform local continuity condition :

$$|p(x)-p(y)|\le\frac{M}{|\log|x-y||},\text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y|<\frac{1}{2},\ M>0. \tag{2.2}$$

We denote $\|.\|_q$ and $\|.\|_{H^1(\Omega)}$ to the usual $L^q(\Omega)$ norm and $H^1(\Omega)$ norm, respectively. To achieve our result, we need the following Lemma:

Lemma 2.1. *([14])*

(1) *If*

$$2\le p^-:=\operatorname*{ess\,inf}_{x\in\Omega}p(x)\le p(x)\le p^+:=\operatorname*{ess\,sup}_{x\in\Omega}p(x)<\infty,$$

then

$$\min\left\{\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right\}\le\int_\Omega|u(.)|^{p(x)}\mathrm{d}x\le\max\left\{\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right\}$$

for any $u\in L^{p(.)}(\Omega)$.

(2) *Let $p,q,r:\overline{\Omega}\to(1,+\infty)$ be measurable functions such that*

$$\frac{1}{p(.)}=\frac{1}{r(.)}+\frac{1}{q(.)}.$$

Then, for all functions $u\in L^{r(.)}(\Omega)$ and $v\in L^{q(.)}(\Omega)$, we have $uv\in L^{p(.)}(\Omega)$ with

$$\|uv\|_{p(.)}\le\mathcal{C}\,\|u\|_{r(.)}\,\|v\|_{q(.)}\,.$$

(3) *If $p:\Omega\to[p^-,p^+]\subset[1,+\infty)$ is a measurable function and $p_*>ess\ \operatorname*{sup}_{\{x\in\Omega\}}p(x)$ with $p_*\le\frac{2n}{n-2}$, then the embedding $H_0^1(\Omega)=W_0^{1,2}(\Omega)\hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact, there is a constant $c_*=c_*(\Omega,p^\pm)$ such that*

$$\|u\|_{p(.)}\le c_*\|\nabla u\|_2 \text{ for } u\in H_0^1(\Omega).$$

For the relaxation function g and the potential α, we have the following assumptions

Hypothesis g, α: g, $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$g(s) \geq 0,\ l_0 = \int_0^\infty g(s)\mathrm{d}s < \infty,\ \alpha(t) > 0,\ a - \alpha(t)\int_0^t g(s)\mathrm{d}s \geq l > 0. \tag{\textbf{H1}}$$

Hypothesis ξ: There exist a positive differentiable functions ξ satisfying

$$g'(t) \leq -\xi(t)g(t),\ \text{for } t \geq 0,\ \lim_{t\to\infty}\frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0. \tag{\textbf{H2}}$$

Hypothesis $p(.)$: The function $p(.)$ satisfies

$$p^- \geq 2, \text{ if } n = 1, 2,\ 2 < p^- \leq p(x) \leq p^+ < \frac{n+2}{n-2} \text{ if } n \geq 3. \tag{\textbf{H3}}$$

Hypothesis μ_1 and μ_2 : The constants μ_1 and μ_2 satisfy

$$|\mu_2| < p^-\mu_1. \tag{\textbf{H4}}$$

it is easy, by differentiating the term $\alpha(t)(g \circ u)(t)$ with respect to t, to show that

$$\begin{aligned}
&\alpha(t)\int_0^t g(t-s)\int_\Omega u(s)\mathrm{d}s u_t(t)\mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\alpha(t)}{2}(g\circ u)(t) - \frac{\alpha(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s\right] - \frac{\alpha(t)}{2}g(t)\|u(t)\|_2^2 \\
&+ \frac{\alpha(t)}{2}(g'\circ u)(t) + \frac{\alpha'(t)}{2}(g\circ u)(t) - \frac{\alpha'(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s,
\end{aligned} \tag{2.7}$$

where

$$(g\circ u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|_2^2\mathrm{d}s.$$

In order to deal with the delay feedback term, motivated by [12, 11], we introduce the following new dependent variable,

$$z(x,\rho,t) = u_t(x, t-\tau\rho), x \in \Omega,\ \rho \in (0,1), t > 0. \tag{2.8}$$

By computation we have

$$\tau z_t(x,\rho,t) + z_p(x,\rho,t) = 0,\ \text{in } \Omega \times (0,1) \times (0,\infty).$$

Therefore, problem (1.1) can be transformed into

$$\begin{aligned}
&u_{tt} - \mathcal{M}\left(|\nabla u(t)|^2\right)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s \\
&+ \mu_1|u_t|^{p(x)-2}u_t + \mu_2|z(1,t)|^{p(x)-2}z(1,t) = 0 \text{ in } \Omega\times\mathbb{R}^+, \\
&\tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \text{ in } (0,1)\times(0,\infty), \\
&z(0,t) = u_t, \text{ in } (0,+\infty), \\
&z(\rho,0) = j_0(-\rho\tau), \text{ in } (0,1), \\
&u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x),\ x \in \Omega.
\end{aligned} \tag{2.9}$$

Relying on Faedo–Galerkin method, we can give the following well-posedness theorem

Theorem 2.2. *Let (H1)-(H4) hold. Then, for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $j_0 \in L^2((\Omega) \times (0,1))$, *there exists a unique local solution* u *of the problem* (1.1) *in the class*

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right),\ u_t \in C\left([0,T]; H_0^1(\Omega)\right) \cap L^2([0,T] \times (\Omega)).$$

3. Main asymptotic theorem

The purpose of this paper is to give a theorem that concerns the asymptotic stability of solutions for the problem (1.1). For this aim, we need the following technical Lemmas

3.1. Technical lemmas.

In this subsection, we present for rather technical Lemmas that we need to the proof of Theorem (3.8). Let define the modified energy functional E associated with problem (2.9) by

$$\begin{aligned} E(t) =& \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{1}{2}\left(a - \alpha(t)\int_0^t g(s)\,\mathrm{d}s\right)\left\|\nabla u\right\|_2^2 + \frac{b}{4}\left\|\nabla u\right\|_2^4 \\ &+ \xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}\left|u_t(x,s)\right|^{p(x)}\mathrm{d}s\mathrm{d}x + \frac{1}{2}\alpha(t)\left(g \circ \nabla(u)\right)(t), \end{aligned} \tag{3.1}$$

where ξ and λ are positive constants in which

$$\mu_1 p^- - |\mu_2| > \xi > |\mu_2|\, p^+ \frac{p^+ - 1}{p^-},\ \lambda < \frac{1}{\tau_1}\left|\ln\frac{\mu_2 p^+\left(p^+ - 1\right)}{\xi p^-}\right|. \tag{3.2}$$

Let us check the following three Lemmas, which are essential to prove the main result given in Theorem (3.8)

Lemma 3.1. *Let u be a solution of problem* (2.9). *Then,*

$$\begin{aligned} E'(t) \le & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{1}{2}\alpha(t)\left(g' \circ \nabla u\right)(t) - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds \\ & - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 + \frac{1}{2}\alpha'(t)(g \circ \nabla u)(t) - \left(\mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x \\ & - \left(\frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\,\frac{p^+ - 1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x \\ & - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}\left|u_t(x,s)\right|^{p(x)}\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.3}$$

Proof. Multiplying the first equation in (2.9) by u_t, integrating over Ω, and multiplying the second equation by $\zeta z e^{-\lambda\tau\rho}$ and integrating over $(0,1)\times\Omega$ with respect to ρ and x, after summing them up we obtain

$$\begin{aligned} E'(t) =& -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2+\frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s \\ &-\frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2+\frac{\alpha'(t)}{2}(g\circ\nabla u)(t)-\mu_1\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x \\ &-\xi\int_\Omega\frac{1}{p\left(x\right)}e^{-\lambda\tau}\left|u_t(x,t-\tau)\right|^{p(x)}\mathrm{d}x-\mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x \\ &+\xi\int_\Omega\frac{1}{p\left(x\right)}\left|u_t(x,t)\right|^{p(x)}\mathrm{d}x-\lambda\xi\int_\Omega\frac{1}{p\left(x\right)}\int_{t-\tau}^t e^{\lambda(s-t)}\left|u_t(x,s)\right|^{p(x)}\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.4}$$

Using Young's inequality and the fact that $z(1,t)=u_t\left(t-\tau\right)$, we get

$$-\mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x\le|\mu_2|\,\frac{p^+-1}{p^-}\int_\Omega |z(1,t)|^{p(x)}dx+\frac{|\mu_2|}{p^-}\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x. \tag{3.5}$$

By (2.8), we have

$$-\xi\int_\Omega\frac{1}{p\left(x\right)}e^{-\lambda\tau}\left|u_t(x,t-\tau)\right|^{p(x)}\mathrm{d}x\le-\frac{\xi}{p^+}e^{-\lambda\tau}\int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x.$$

Combining (3.4), and (3.5), we obtain

$$\begin{aligned} E'(t)\le& -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2+\frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s \\ &-\frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2+\frac{\alpha'(t)}{2}(g\circ\nabla u)(t)-\left(\mu_1-\frac{\xi}{p^-}-\frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x. \\ &-\left(\frac{\xi}{p^+}e^{-\lambda\tau}-|\mu_2|\,\frac{p^+-1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x \\ &-\lambda\xi\int_\Omega\frac{1}{p\left(x\right)}\int_{t-\tau}^t e^{\lambda(s-t)}\left|u_t(x,s)\right|^{p(x)}\mathrm{d}s\mathrm{d}x. \end{aligned}$$

Letting

$$c_0=\mu_1-\frac{\xi}{p^-}-\frac{|\mu_2|}{p^-}\text{ and }c_1=\frac{\xi}{p^+}e^{-\lambda\tau}-|\mu_2|\,\frac{p^+-1}{p^-},$$

and using condition (3.2), we get the desired inequality (3.3). □

Remark 3.2. *As* $-\frac{1}{2}\alpha'\left(t\right)\|\nabla u(t)\|_2^2\int_0^t g\left(s\right)\mathrm{d}s\ge 0$, $E(t)$ *may not be non-increasing.*

Lemma 3.3. *Let* u *be a solution of problem* (2.9)*. Then,*

$$\|\nabla u\|_2^2\le\frac{2E(0)}{l}e^{\frac{l_0}{l}\alpha(0)},\ t\ge 0. \tag{3.6}$$

Proof. From (3.3), and (3.1) we have

$$E'(t)\le-\frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds\le-\frac{1}{2}l_0\alpha'(t)\|\nabla u\|_2^2\le-\frac{l_0}{l}\alpha'(t)E(t).$$

Integrating this over $(0,t)$, we conclude that

$$E(t)\le E(0)e^{-\frac{l_0}{l}\alpha(t)+\frac{l_0}{l}\alpha(0)}\le E(0)e^{\frac{l_0}{l}\alpha(0)},$$

consequently (3.6) remains valid. □

Now, let us define the modified energy by

$$\mathrm{L}(t) = NE(t) + \varepsilon_1\alpha(t)\varphi(t) + \varepsilon_2\alpha(t)\psi(t), \tag{3.7}$$

in which ε_1 and ε_2 are positive constants to be chosen later, and

$$\varphi(t) = \int_\Omega u(t)u_t(t)\mathrm{d}x + \frac{\sigma}{4}\|\nabla u\|_2^4, \tag{3.8}$$

$$\psi(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\mathrm{d}x. \tag{3.9}$$

The functional L is equivalent to the energy function E by the following lemma.

Lemma 3.4. *There exist two positive constants C_1 and C_2 such that,*

$$C_1E(t) \le \mathrm{L}(t) \le C_2E(t), \;\; t \ge 0. \tag{3.10}$$

Proof. Integrating by parts, using Young's inequality and Poincare's Theorem, we have

$$\begin{aligned}
|\mathrm{L}(t) - NE(t)| &= \left|\alpha(t)\int_\Omega u(t)u_t(t)\mathrm{d}x + \alpha(t)\frac{\sigma}{4}\|\nabla u\|_2^4\right| \\
&\le \varepsilon_1|\alpha(t)|\int_\Omega |u(t)||u_t(t)|\,\mathrm{d}x + \varepsilon_1\frac{\sigma}{4}|\alpha(t)|\,\|\nabla u\|_2^4 \\
&+ \varepsilon_2\frac{1}{2}|\alpha(t)|\,\|u_t\|_2^2 + \varepsilon_2\frac{1}{2}|\alpha(t)|\,c_*^2(a-l)(g\circ\nabla(u))(t) \\
&\le \frac{\alpha(0)}{2}c_*^2\|\nabla u\|_2^2 + \frac{\alpha(0)}{2}\|u_t\|_2^2 + \sigma\frac{\alpha(0)}{4}\|\nabla u\|_2^4 \\
&+ \varepsilon_2\frac{1}{2}\alpha(0)\|u_t\|_2^2 + \varepsilon_2\frac{1}{2}\alpha(0)\,c_*^2(a-l)(g\circ\nabla(u))(t) \\
&\le C(\varepsilon_1+\varepsilon_2)E(t),
\end{aligned}$$

taking $C_1 = N - C(\varepsilon_1+\varepsilon_2)$ and $C_2 = N + C(\varepsilon_1+\varepsilon_2)$, with ε_1, and ε_2 sufficiently small, the proof of Lemma (3.4) is concludes. □

Lemma 3.5. *There exist positive constants c_ε and C_ε satisfying*

$$\begin{aligned}
\varphi'(t) &\le \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\frac{a}{2l}(g\circ\nabla u)(t) \\
&\quad + c_\varepsilon\left(\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}dx\right) + C_\varepsilon\int_\Omega |u|^{p(x)}\,\mathrm{d}x.
\end{aligned} \tag{3.11}$$

Proof. Differentiating (3.8) with respect to t, using first equation of (2.9), we get

$$\begin{aligned}
\varphi'(t) &= \|u_t\|_2^2 + \int_\Omega uu_{tt}\mathrm{d}x + \sigma\|\nabla u\|_2^2\int_\Omega \nabla u\nabla u_t\mathrm{d}x + \int_\Omega |u|^{p(x)-2}uu_t\mathrm{d}x \\
&= \|u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u(t)\mathrm{d}x \\
&- \mu_1\int_\Omega |u_t|^{p(x)-2}u_tu\mathrm{d}x - \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u\mathrm{d}x + \int_\Omega |u|^{p(x)-2}uu_t\mathrm{d}x \\
&= \|u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.12}$$

We need to estimate the second summand on the right-hand side of (3.12). Hölder inequality, Young's inequality, Sobolev-Poincare inequalities, (H1), and (3.3), give

$$
\begin{aligned}
I_1 =&\alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\,(t)\,\mathrm{d}x\\
\leq&\alpha(t)\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
\leq&\alpha(t)\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega\int_0^t g(s)\mathrm{d}s\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{\frac{1}{2}}\\
\leq&\alpha(t)\left(\int_\Omega|\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{\frac{1}{2}}\\
\leq&\frac{\alpha(t)}{2}\int_\Omega|\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s+\frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\\
\leq&\frac{\alpha(t)}{2}\int_\Omega|\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s+\frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)+\nabla u(t)|^2\mathrm{d}s\mathrm{d}x,
\end{aligned}
\tag{3.13}
$$

we use Young's inequality, and (H1) to obtain for every $\eta>0$

$$
\begin{aligned}
&\frac{\alpha\,(t)}{2}\int_\Omega\int_0^t g(t-s)[\nabla u(s)-\nabla u(t)+\nabla u(t)]^2\mathrm{d}s\mathrm{d}x\\
\leq&\frac{\alpha\,(t)}{2}\int_\Omega\int_0^t g(t-s)\left((\nabla u(s)-\nabla u(t))^2+2|\nabla u(s)-\nabla u(t)||\nabla u|+|\nabla u|^2\right)\mathrm{d}s\mathrm{d}x\\
\leq&\frac{\alpha\,(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\mathrm{d}s\mathrm{d}x+\frac{\alpha\,(t)}{2}\int_a\int_0^t g(t-s)|\nabla u|^2\mathrm{d}s\mathrm{d}x\\
&+\alpha\,(t)\int_\Omega\int_0^t g(t-s)\,|\nabla u(s)-\nabla u(t)|\,|\nabla u|\,\mathrm{d}s\mathrm{d}x\\
\leq&\frac{\alpha\,(t)}{2}(g\circ\nabla u)(t)+\frac{\alpha\,(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega|\nabla u|^2\mathrm{d}x\\
&+\eta\frac{\alpha\,(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{\alpha\,(t)}{2\eta}(g\circ\nabla u)(t)\\
\leq&\frac{\alpha\,(t)}{2}(1+\eta)\int_0^t g(s)\mathrm{d}s\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{\alpha\,(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t)\\
\leq&(1+\eta)\frac{(a-l)}{2}\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{\alpha\,(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t).
\end{aligned}
\tag{3.14}
$$

By combining (3.13) and (3.14) we arrive at

$$
\begin{aligned}
&\alpha\,(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x\\
\leq&\frac{(a-l)}{2}\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{(a-l)}{2}(1+\eta)\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{\alpha\,(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t)\\
=&(2+\eta)\frac{(a-l)}{2}\|\nabla u\|^2+\frac{\alpha\,(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t).
\end{aligned}
$$

By taking $\eta = \frac{l}{a-l}$, we have

$$\begin{aligned} |I_1| \leq& \alpha(t) \int_\Omega \int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x \\ \leq& \left(a - \frac{l}{2}\right) \|\nabla u\|_2^2 + \frac{a}{2l}\alpha(t)(g \circ \nabla u)(t). \end{aligned} \tag{3.15}$$

and by using Young's inequality

$$\begin{aligned} |I_2| \leq& c_\varepsilon \int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \varepsilon \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \int_\Omega |u|^{p(x)}\,\mathrm{d}x \\ :=& c_\varepsilon \int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \varepsilon c_2 \int_\Omega |u|^{p(x)}\,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Related computations further allow,

$$\begin{aligned} |I_3| \leq& c_\varepsilon \int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x + \varepsilon \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \int_\Omega |u|^{p(x)}\,\mathrm{d}x \\ :=& c_\varepsilon \int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x + \varepsilon c_3 \int_\Omega |u|^{p(x)}\,\mathrm{d}x, \end{aligned} \tag{3.17}$$

$$I_4 \leq c_\varepsilon \int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \varepsilon c_4 \int_\Omega |u|^{p(x)}\,\mathrm{d}x$$

A substitution of (3.15)-(3.17) into (3.12), we get

$$\begin{aligned} \varphi'(t) \leq& \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 + C_\varepsilon \int_\Omega |u|^{p(x)}\,\mathrm{d}x - b\|\nabla u\|_2^4 \\ &+ \frac{a}{2l}\alpha(t)(g \circ \nabla u)(t) + c_\varepsilon \left(\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x\right), \end{aligned} \tag{3.18}$$

where, for ε sufficiently small, $C_\varepsilon = \varepsilon(c_2 + c_3 + c_4) > 0$. □

Lemma 3.6. *There exist positive constants δ and c_δ satisfying*

$$\begin{aligned} \psi'(t) \leq& -\left\{\left(\int_0^t g(s)ds\right\} - \delta\right) \|u_t\|_2^2 + \delta\left\{a + 2(a-l)^2\alpha(t)\right\} \|\nabla u\|_2^2 + \delta b\|\nabla u\|_2^4 \\ &+ \delta\frac{2\sigma E(0)}{l} e^{\frac{l_0}{l}\alpha(0)} \left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right)(a-l)\alpha(t)\right\}(g \circ \nabla u)(t) \\ &+ c_\delta \left(\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x\right) - \frac{g(0)c_*^2}{4\delta}\left(g' \circ \nabla u\right)(t). \end{aligned} \tag{3.19}$$

Proof. Differentiating (3.9) with respect to t and using first equation of (2.9), yields

$$
\begin{aligned}
\psi'(t) =& -\int_\Omega u_{tt} \int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
& - \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s \ \mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\
& = \left(a + b\|\nabla u\|_2^2\right) \int_\Omega \nabla u \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\mathrm{d}s\mathrm{d}x \\
& + \sigma \int_\Omega \nabla u \nabla u_t \ \mathrm{d}x \int_\Omega \nabla u \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\mathrm{d}s\mathrm{d}x \\
& - \alpha(t) \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\
& + \mu_1 \int_\Omega |u_t|^{p(x)-2} u_t \int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
& + \mu_2 \int_\Omega |z(1,t)|^{p(x)-2} z(1,t) \int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
& - \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\
& = \sum_{i=1}^{6} I_i - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2,
\end{aligned}
\tag{3.20}
$$

in what follows, we need to estimate the second summand on the right-hand side of (3.20). By using Hölder's, Young's, Sobolev-Poincare inequalities, (H1), (3.6) and (3.3), we obtain

$$
\begin{aligned}
|I_1| \le& \left(a + b\|\nabla u\|_2^2\right)\left\{\delta\|\nabla u\|_2^2 + \frac{(a-l)}{4\delta}(g \circ \nabla u)(t)\right\} \\
\le& \delta a\|\nabla u\|_2^2 + \delta b\|\nabla u\|_2^4 + \left\{\frac{a(a-l)}{4\delta} + \frac{b(a-l)E(0)}{2\delta l} e^{\frac{l_0}{l}\alpha(0)}\right\}(g \circ \nabla u)(t),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
|I_2| \le& \delta\sigma \left(\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right)^2 \|\nabla u\|_2^2 + \frac{\sigma(a-l)}{4\delta}(g \circ \nabla u)(t) \\
\le& \delta \frac{2\sigma E(0)}{l} e^{\frac{l_0}{l}\alpha(0)} \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 + \frac{\sigma(a-l)}{4\delta}(g \circ \nabla u)(t),
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
|I_3| \le& \delta\alpha(t) \int_\Omega \left(\int_0^t g(t-s)(|\nabla u(t) - \nabla u(s)| + |\nabla u(t)|)\mathrm{d}s\right)^2 \mathrm{d}x \\
& + \frac{1}{4\delta}\alpha(t) \int_\Omega \left(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|\mathrm{d}s\right)^2 \mathrm{d}x \\
\le& 2\delta(a-l)^2\alpha(t)\|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)(a-l)\alpha(t)(g \circ \nabla u)(t),
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
|I_4| \leq & c_\delta \int_\Omega |u_t|^{p(x)} \,\mathrm{d}x + \delta \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^{p(x)} \mathrm{d}x \\
& \leq c_\delta \int_\Omega |u_t|^{p(x)} \,\mathrm{d}x + \delta \left\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left((a-l)^{p^+-1}, (a-l)^{p^--1}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right)\right. \\
& \left.\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^{p(x)}\mathrm{d}s\right\} \\
& \leq c_\delta \int_\Omega |u_t|^{p(x)} \,\mathrm{d}x + \delta \left\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right)\right. \\
& \left.\max\left(\left(\frac{2E(0)}{l} e^{\frac{l_0}{l}\alpha(0)}\right)^{(p^+-2)/2}, \left(\frac{2E(0)}{l} e^{\frac{l_0}{l}\alpha(0)}\right)^{(p^--2)/2}\right)(g \circ \nabla u)(t)\right\} \\
& := c_\delta \int_\Omega |u_t|^{p(x)} \,\mathrm{d}x + \delta c_4 (g \circ \nabla u)(t).
\end{aligned} \tag{3.24}
$$

Similarly

$$
\begin{aligned}
|I_5| &\leq c_\delta \int_\Omega |z(1,t)|^{p(x)} \,\mathrm{d}x + \delta c_5 (g \circ \nabla u)(t), \\
|I_6| &\leq \delta \|u_t\|_2^2 - \frac{g(0)c_*^2}{4\delta} \left(g' \circ \nabla u\right)(t).
\end{aligned} \tag{3.25}
$$

Combining these estimates (3.21)-(3.25) and then (3.20) becomes

$$
\begin{aligned}
\psi'(t) \leq & -\left(\int_0^t g(s)\mathrm{d}s - \delta\right) \|u_t\|_2^2 + \delta \left\{a + 2(a-l)^2\alpha(t)\right\} \|\nabla u\|_2^2 + \delta b \|\nabla u\|_2^4 \\
& + \delta \frac{2\sigma E(0)}{l} e^{\frac{l_0}{l}\alpha(0)} \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 + \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right)(a-l)\alpha(t)\right\}(g \circ \nabla u)(t) \\
& + c_\delta \left(\int_\Omega |u_t|^{p(x)} \,\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)} \,\mathrm{d}x\right) - \frac{g(0)c_*^2}{4\delta} \left(g' \circ \nabla u\right)(t),
\end{aligned} \tag{3.26}
$$

in which $C_\delta = \left\{\frac{a(a-l)}{4\delta} + \frac{b(a-l)E(0)}{2\delta l} e^{\frac{l_0}{l}\alpha(0)} + \frac{\sigma(a-l)}{4\delta} + \delta\left(c_4 + c_5\right)\right\}$. □

Lemma 3.7. *There exist positive constants* C_3, C_4 *and* t_0 *satisfying*

$$
\mathrm{L}'(t) \leq -\mathrm{C}_3 \alpha(t) E(t) + \mathrm{C}_4 \alpha(t)(g \circ \nabla u)(t), \quad t > t_0. \tag{3.27}
$$

Proof. Since the function g is positive, continuous and $g(0) > 0$, then for any $t \geq t_0 > 0$, we have

$$
\int_0^t g(s)\mathrm{d}s \geq \int_0^{t_0} g(s)\mathrm{d}s = g_0 > 0.
$$

Differentiating (3.9), and using Lemmas (3.5), (3.5), we have

$$
\begin{aligned}
\mathrm{L}'(t) =& NE'(t) + \varepsilon_1 \alpha'(t)\varphi(t) + \varepsilon_1\alpha(t)\varphi'(t) + \varepsilon_2\alpha'(t)\psi(t) + \varepsilon_2\alpha(t)\psi'(t) \\
\leq& -\alpha(t)\left\{\varepsilon_2\left(g_0-\delta\right)-\varepsilon_1\right\}\|u_t\|_2^2 - \alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a+2(a-l)^2\right)\alpha(t)\right\}\|\nabla u\|_2^2 \\
&-\alpha(t)\left(b\left(\varepsilon_1-\varepsilon_2\delta\right)\right)\|u_t\|_2^4 - \alpha(t)\left\{\sigma - \varepsilon_2\delta\frac{\sigma E(0)}{l}e^{\frac{l_0}{l}\alpha(0)}\right\}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \\
&+\alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta+\frac{1}{4\delta}\right)(a-l)\alpha(t)\right\}(g\circ\nabla u)(t) \\
&+\alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t) - \alpha(t)\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x \\
&-\alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |z(1,t)|^{p(x)} - \frac{N\alpha'(t)}{2}\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\
&+\varepsilon_1\alpha'(t)\int_\Omega u u_t\mathrm{d}x + \varepsilon_2\alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{3.28}
$$

Making use of the following relations

$$
\begin{aligned}
\alpha'(t)\int_\Omega u u_t\mathrm{d}x + &\alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
&\leq -\alpha'(t)\frac{c_*^2}{2}\|\nabla u\|_2^2 - \alpha'(t)\|u_t\|_2^2 - \alpha'(t)\frac{c_*^2}{2}\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{L}'(t) \leq& -\alpha(t)\left\{\varepsilon_2\left(g_0-\delta\right)-\varepsilon_1+\frac{\alpha'(t)}{\alpha(t)}\right\}\|u_t\|_2^2 \\
&-\alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a+2(a-l)^2\right)\alpha(0) + \frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right) + \frac{c_*^2\alpha'(t)}{2\alpha(t)}\right\}\|\nabla u\|_2^2 \\
&-\alpha(t)\left(b\left(\varepsilon_1-\varepsilon_2\delta\right)\right)\|\nabla u\|_2^4 - \alpha(t)\left\{\sigma\varepsilon_1 - \varepsilon_2\delta\frac{\sigma E(0)}{l}e^{\frac{l_0}{l}\alpha(0)}\right\}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 \\
&+\alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta+\frac{1}{4\delta}\right)(a-l)\alpha(t) - \frac{c_*^2\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right)\right\}(g\circ\nabla u)(t) \\
&+\alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t) - \alpha(t)\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\,\mathrm{d}x \\
&-\alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |z(1,t)|^{p(x)}\,\mathrm{d}x.
\end{aligned} \tag{3.29}
$$

First, we fix $\delta > 0$ such that,

$$
g_0 - \delta > \frac{1}{2}g_0,\ \frac{\delta}{C_\varepsilon}\left(a+2(a-l)^2\right)\alpha(0) < \frac{1}{4}g_0,
$$

and take ε_1 and ε_2 so small satisfying

$$
\frac{g_0}{4}\varepsilon_2 < \varepsilon_1 < \varepsilon_2\frac{g_0}{2}, \tag{3.30}
$$

and

$$
c_5 = \varepsilon_2\left(g_0-\delta\right) - \varepsilon_1 > 0,
$$

$$c_6 = \varepsilon_1 C_\varepsilon - \varepsilon_2 \delta \left(a + 2(a-l)^2\right) \alpha(0) > 0.$$

We select ε_1 and ε_2 small enough so that relations (3.10) and (3.29) are valid, furthermore

$$b\left(\varepsilon_1 - \varepsilon_2 \delta\right) > 0,\ \sigma\varepsilon_1 - \varepsilon_2 \delta \frac{\sigma E(0)}{l} e^{\frac{l_0}{l}\alpha(0)} > 0,\ \frac{N}{2} - \varepsilon_2 \frac{g(0)c_*^2}{4\delta} > 0$$
$$\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta > 0,\ \frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta > 0.$$

Hence, (3.29) becomes, for a generic positive constant c

$$\begin{aligned} \mathrm{L}'(t) \leq & -\alpha(t)\left\{c + \frac{\alpha'(t)}{\alpha(t)}\right\} \|u_t\|_2^2 - \alpha(t)\left\{c + \frac{\alpha'(t)}{2\alpha(t)}\left(\left(\int_0^t g(s)\mathrm{d}s\right) + \frac{c_*^2}{2}\right)\right\} \|\nabla u\|_2^2 \\ & + \alpha(t)\left\{c - \frac{c_*^2 g_0 \alpha'(t)}{2\alpha(t)}\right\}(g \circ \nabla u)(t),\ \forall t \geq t_0. \end{aligned} \tag{3.31}$$

Since $\lim\limits_{t\to\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0$, we can pick $t_1 > t_0$ so that (3.31) leads to

$$\begin{aligned} \mathrm{L}'(t) \leq & -\alpha(t)\left(c\,\|u_t\|_2^2 + C\|\nabla u\|_2^2\right) + c(g \circ \nabla u)(t) \\ \leq & -\mathrm{C}_3 \alpha(t) E(t) + \mathrm{C}_4 \alpha(t)(g \circ \nabla u)(t),\ \forall t \geq t_1, \end{aligned} \tag{3.32}$$

where C_3 and C_4 are positive constants. □

Next, the main result reads as follows

Theorem 3.8. *Assume* (H1)*-*(H4) *and* (3.2) *hold. Then there exist positive constants* C_0*,* C*, and* $t_1 > 0$ *such that the energy for problem* (1.1) *satisfy*

$$E(t) \leq \mathrm{C}_0 e^{-\mathrm{C}\int_{t_1}^t \alpha(s)\gamma(s)\mathrm{d}s},\ \textit{for } t \geq t_1.$$

Proof of Theorem (3.8). By using Lemma (3.7), (3.1) and (H1), we get

$$\begin{aligned} \zeta(t)\mathrm{L}'(t) & \leq -\mathrm{C}_3\alpha(t)\zeta(t)E(t) + \mathrm{C}_4\alpha(t)\zeta(t)(g \circ \nabla u)(t) \\ & \leq -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \mathrm{C}_4\alpha(t)\left(g' \circ \nabla u\right)(t) \\ & \leq -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \mathrm{C}_4\left(2E'(t) + \alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\right). \end{aligned} \tag{3.33}$$

Since $\zeta(t)$ is nonincreasing, the definition of $E(t)$ and and assumption (H1), we have

$$\frac{l}{2}\|\nabla u\|_2^2 \leq E(t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\zeta(t)\mathrm{L}(t) + 2\mathrm{C}_4E(t)\right) \leq -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \mathrm{C}_4\alpha'(t)\left(\int_0^s g(s)\mathrm{d}s\right)\|\nabla u\|_2^2, \tag{3.34}$$

which leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\zeta(t)F(t) + 2\mathrm{C}_4E(t)\right) \leq & -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \mathrm{C}_4\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\ \leq & -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \frac{2\mathrm{C}_4E(t)}{l}\alpha'(t)\int_0^t g(s)\mathrm{d}s \\ \leq & -\alpha(t)\zeta(t)\left(\mathrm{C}_3 + \frac{2\mathrm{C}_4 l_0 \alpha'(t)}{l\alpha(t)\zeta(t)}\right)E(t). \end{aligned} \tag{3.35}$$

Since $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)\zeta(t)} = 0$, we can choose $t_1 \geq t_0$ such that $\mathrm{C}_3 + \frac{2\mathrm{C}_4 l_0 \alpha'(t)}{l\alpha(t)\zeta(t)} > 0$ for $t \geq t_1$. Hence, if we let

$$\mathcal{L}(t) = \zeta(t)\mathrm{L}(t) + 2\mathrm{C}_4 E(t)$$

easily we see that $\mathcal{L}(t)$ is equivalent to $E(t)$, and satisfy

$$\mathcal{L}'(t) \leq -k\zeta(t)\alpha(t)\mathcal{L}(t) \text{ for } t \geq t_1. \tag{3.36}$$

Simple integrating (3.36) over (t_1, t). with respect to t, yields

$$\mathcal{L}(t) \leq \mathcal{L}(t_1)\, e^{-\mathrm{C}\int_{t_1}^{t} \zeta(s)\alpha(s)\mathrm{d}s} t \geq t_0.$$

Thus, the desired result yields from the equivalence relations of $\mathcal{L}(t)$, $\mathrm{L}(t)$, and $E(t)$. □

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5. Data Availability

No data is used in the manuscript.

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