Stability and large-time behavior of the 2D Boussinesq system with mixed partial dissipations near hydrostatic equilibrium

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Abstract

The purpose of this note is to address the stability and large-time behavior for the 2D Boussinesq system with vertical dissipation on \$u_1\$ and horizontal dissipation on \$u_2\$ near a hydrostatic equilibrium. Meanwhile the decay estimates of that system are also presented. Finally, we also obtain the decay rates of the solution to the corresponding linearized equation of the Boussinesq system.

STABILITY AND LARGE-TIME BEHAVIOR OF THE 2D BOUSSINESQ SYSTEM WITH MIXED PARTIAL DISSIPATIONS NEAR HYDROSTATIC EQUILIBRIUM

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Keywords: Boussinesq equations, hydrostatic equilibrium, stability, large-time behavior.

Mathematics Subject Classifications (2010): 35Q30, 76D03, 76D07.

1. Introduction

This paper aims to investigate the following two dimensional Boussinesq equations with partial dissipation

$$
\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 - \nu\partial_{22}u_1 + \partial_1\Pi = 0, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \partial_t u_2 + (u\cdot\nabla)u_2 - \nu\partial_{11}u_2 + \partial_2\Pi = \Theta, \\ \partial_t\theta + (u\cdot\nabla)\Theta = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}
$$

where $u = (u_1, u_2)$ is the velocity field, Θ, Π denote the temperature and the pressure, respectively. The positive constant ν is the viscosity. Obviously, the Boussinesq system (1.1) has a steady state solution

$$
u^0 = (0,0), \quad \Theta^0 = x_2, \quad \Pi^0 = \frac{1}{2}x_2^2, \tag{1.2}
$$

which is often named the hydrostatic equilibrium. We consider the perturbation (u, θ, π) with

$$
u = u - u^0, \theta = \Theta - \Theta^0, \pi = \Pi - \Pi^0.
$$

Then one can verify that

$$
\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 - \nu\partial_{22}u_1 + \partial_1\pi = 0, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \partial_t u_2 + (u\cdot\nabla)u_2 - \nu\partial_{11}u_2 + \partial_2\pi = \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta = -u_2, \\ \nabla\cdot u = 0. \end{cases} \tag{1.3}
$$

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It is well known that the global well-posedness of this system remain open. If we add an damping term into the temperature equation, the system (1.3) becomes

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 - \nu\partial_{22}u_1 + \partial_1\pi = 0, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \partial_t u_2 + (u\cdot\nabla)u_2 - \nu\partial_{11}u_2 + \partial_2\pi = \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta + \eta\theta = -u_2, \\ \nabla\cdot u = 0. \end{cases} \tag{1.4}$$

It is well known that classic Boussinesq equations model buoyancy drift fluids such as atmospherical and oceanographic flows(see e. g. [12], [13], [14]). In addition to natural sciences, the Boussinesq flows usually appears in industrial applications such as dense gas dispersion and central heating. The 2D incompressible Boussinesq system is one of the most commonly studied models in mathematical fluid dynamics. One of its characteristic feature is that special case of the model can be identified with the 3D incompressible Euler equations for axisymmetric swirling flows. Another important qualitative property is that the 2D Boussinesq equations share a similar vortex stretching effect as in the 3D flows (see [19]).

During the past thirty years, a large amount of attention has been paid to the global regularity and stability problems of the Boussinesq equations. The great advances since then have come in the global regularity of the two dimensional Boussinesq equations with only partial or fractional dissipation or even no dissipation (see e. g. [1], [2], [3], [4], [5], [6]). Though the study on the stability and large time behavior is relatively recent in the last fifteen years, the investigations on those problems have so far been great fruitful (see [7], [8], [17], [18], [9]).

In 2021, Lai, Wu and Zhong [10] has established the global existence and stability of the following 2D Boussinesq equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u_1 - \nu\partial_{22}u + \nabla\pi = \theta e_2, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \eta\theta = -u_2, \\ \nabla\cdot u = 0, \\ (u,\theta)|_{t=0} = (u_0,\theta_0), \end{cases} \tag{1.5}$$

in the Sobolev space H^2. They also obtained the large time behavior of $\|\nabla u(t)\|_{L^2}$ and $\|\nabla\theta(t)\|_{L^2}$ in terms of energy methods. Later Lai, Wu et al [11] acquired the optimal decay estimates for the system (1.5). Motivated by [9], [10], [11]and [18], the purpose of this paper is to address the stability and decay of the Boussinesq system (1.4) near the hydrostatic equilibrium.

Our results can be formulated as follows.

Theorem 1.1. *Let $(u_0,\theta_0)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ and $\nabla\cdot u_0 = 0$. Then there exists a constant $\epsilon > 0$ such that if*

$$\|(u_0,\theta_0)\|_{H^2} \le \epsilon, \tag{1.6}$$

then there admits a unique global solution (u,θ) of system (1.4) such that for any $t > 0$,

$$\|(u,\theta)\|_{H^2}^2 + \int_0^t (\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\theta\|_{H^2}^2)d\tau \le C\epsilon^2, \tag{1.7}$$

where C is a positive constant independent of ϵ and t.

Remark 1.2. *This theorem is obtained heavily based on H^2-energy estimate and the bootstrap argument.*

Theorem 1.3. *Let* $(u_0,\theta_0)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$, $(\partial_{111}u_{20},\partial_{222}u_{10})\in L^2(\mathbb{R}^2), \nabla\cdot u_0=0$ *and*

$$\|(u_0,\theta_0)\|_{H^2}\leq\epsilon \tag{1.8}$$

hold for some sufficiently small $\epsilon>0$. *Suppose* (u,θ) *is the corresponding solution of* (1.4) *obtained in Theorem 1.1, then*

(i) As $t\to\infty$, *it holds*

$$\|\partial_1\nabla u_2\|_{L^2}\to 0,\ \|\partial_2\nabla u_1\|_{L^2}\to 0,\ \|\partial_t u\|_{L^2}\to 0,\ \|\theta\|_{L^2}\to 0,\ \|\nabla^2\theta\|_{L^2}\to 0.$$

(ii)It holds

$$\|\nabla u(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}},\ \|\nabla\theta(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}},$$

where C *is a positive constant independent of* t.

Remark 1.4. *Though reasoning in a similar line in* [10], *we can prove the following result. However, due to lacking the horizontal dissipation on* u_1 *and vertical dissipation on* u_2, *we cannot prove that* $\|\theta\|_{L^2},\|\partial_t\theta\|_{L^2}\to 0$ *as* $t\to\infty$. *Moreover, the pressure term brings more difficulties during handling the asymptotic behave of* $\|\partial_t u_1\|_{L^2}$ *and* $\|\partial_t u_2\|_{L^2}$.

Now we turn to solve the linearized system of (1.4)

$$\begin{cases}\partial_{tt}u_1+(\eta+\nu\mathcal{R}_2^2\partial_2^2+\nu\mathcal{R}_1^2\partial_1^2)\partial_t u_1-(\mathcal{R}_1^2-\nu\eta\mathcal{R}_2^2\partial_2^2-\nu\eta\mathcal{R}_1^2\partial_1^2)u_1=0,\\ \partial_{tt}u_2+(\eta+\nu\mathcal{R}_2^2\partial_2^2+\nu\mathcal{R}_1^2\partial_1^2)\partial_t u_2-(\mathcal{R}_1^2-\nu\eta\mathcal{R}_2^2\partial_2^2-\nu\eta\mathcal{R}_1^2\partial_1^2)u_2=0,\\ \partial_{tt}\theta+(\eta+\nu\mathcal{R}_2^2\partial_2^2+\nu\mathcal{R}_1^2\partial_1^2)\partial_t\theta-(\mathcal{R}_1^2-\nu\eta\mathcal{R}_2^2\partial_2^2-\nu\eta\mathcal{R}_1^2\partial_1^2)\theta=0,\end{cases} \tag{1.9}$$

which is very different from that in [10].

Theorem 1.5. *Assume that* (u_0,θ_0) *is the initial data of* (1.9). *Then the solution of* (1.9) *can be given via* u_{10}, u_{20} *and* θ_0 *as*

$$\begin{aligned}u_{10}&=\frac{1}{2}(\eta-\nu\mathcal{R}_1^2\partial_1^2-\nu\mathcal{R}_2^2\partial_2^2)G_1u_{10}-\partial_1\Delta^{-1}\partial_2G_1\theta_0+G_2u_{10},\\ u_{20}&=\frac{1}{2}(\eta-\nu\mathcal{R}_1^2\partial_1^2-\nu\mathcal{R}_2^2\partial_2^2)G_1u_{20}+\partial_1\Delta^{-1}\partial_1G_1\theta_0+G_2u_{20},\\ \theta_0&=-\frac{1}{2}(\eta-\nu\mathcal{R}_1^2\partial_1^2-\nu\mathcal{R}_2^2\partial_2^2)G_1\theta_0-G_1u_{20}+G_2\theta_0.\end{aligned}$$

Here G_1 *and* G_2 *satisfy*

$$\widehat{G_1}(\xi,t)=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1},\ \widehat{G_2}(\xi,t)=\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t}). \tag{1.10}$$

with λ_1 *and* λ_2 *being the roots of the following equation:*

$$\lambda^2+(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})\lambda+(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})=0$$

or

$$\begin{aligned}\lambda_1&=-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})-\frac{1}{2}\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})},\\ \lambda_2&=-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})+\frac{1}{2}\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})}.\end{aligned}$$

Consequently, if $(u_0,\theta_0)\in L^1\cap L^2$ *and* $\nabla\cdot u_0=0$, *then* (u,θ) *fulfils, for any* $0<\sigma<1$,

$$\|(u,\theta)\|_{L^2}\leq C(\sigma)(1+t)^{-\frac{\sigma}{2}}\|(u_0,\theta_0)\|_{L^1\cap L^2},$$

where $C = C(\sigma)$ is a constant depending on σ.

Remark 1.6. *To obtain the decay rates of L^2-norm of u and θ, the crucial step is to establish the upper bound of G_1 and G_2. Compared with* [10]*, the characteristic values in Theorem 1.5 is more complex. To find the upper bound for G_1 and G_2, it needs more ingenious technique, which is provided in Lemma 5.2.*

The rest of this paper is organized as follows. Some crucial lemmas is presented in Section 2. The proof of Theorem 1.1 can be found in Section 3. We will prove Theorem 1.3 and Theorem 1.5 in Section 4 and Section 5, respectively.

2. Preliminaries

The following lemmas play a crucial role in proving our theorem.

Lemma 2.1. *(See* [10]*) Let $f,\ g,\ h,\ \partial_2 g, \partial_1 h \in L^2(\mathbb{R}^2)$. Then there exists a pure constant $C > 0$ such that*

$$\int_{\mathbb{R}^2} |fgh| dx \le C\|f\|_{L^2(\mathbb{R}^2)} \|g\|^{\frac12}_{L^2(\mathbb{R}^2)} \|\partial_2 g\|^{\frac12}_{L^2(\mathbb{R}^2)} \|h\|^{\frac12}_{L^2(\mathbb{R}^2)} \|\partial_1 h\|^{\frac12}_{L^2(\mathbb{R}^2)}.$$

Lemma 2.2. *(See* [10]*) It holds that*

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|^{\frac14}_{L^2(\mathbb{R}^2)} \|\partial_1 f\|^{\frac14}_{L^2(\mathbb{R}^2)} \|\partial_2 f\|^{\frac14}_{L^2(\mathbb{R}^2)} \|\partial_{12}\|^{\frac14}_{L^2(\mathbb{R}^2)}.$$

when the right sides are all bounded. Consequently, the following inequalities hold

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|^{\frac12}_{H^1(\mathbb{R}^2)} \|\partial_1 f\|^{\frac12}_{H^1(\mathbb{R}^2)},$$

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|^{\frac12}_{H^1(\mathbb{R}^2)} \|\partial_2 f\|^{\frac12}_{H^1(\mathbb{R}^2)},$$

when the right sides are all bounded.

Lemma 2.3. *(See* [10]*) Suppose $f = f(t)$ is a nonnegative continuous function for $t \in [0, \infty)$. Let f be integrable on $[0, \infty)$,*

$$\int_0^\infty f(t) dt < \infty.$$

Assume that for any $\delta > 0$, there is $\rho > 0$ such that, for any $0 \le t_1 < t_2$ with $t_2 - t_1 \le \rho$,

$$\text{either } f(t_2) \le f(t_1) \text{ or } f(t_2) \ge f(t_1) \text{ and } f(t_2) - f(t_1) \le \delta.$$

Then

$$f(t) \to 0 \text{ as } t \to \infty.$$

Lemma 2.4. *(See* [10]*) Suppose $f \in W^{1,1}([0, \infty))$, that is*

$$\int_0^\infty |f(t)| dt < \infty \text{ and } \int_0^\infty |f'(t)| dt < \infty.$$

Then $f(t) \to 0$ as $t \to \infty$.

Lemma 2.5. *(See* [10]*) Let $f(t)$ be a nonnegative function satisfying for two constant $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty |f(\tau)| d\tau \le C_0 < \infty \text{ and } f(t) \le C_1 f(s) \text{ for any } 0 \le s < t.$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0 C_1\}$ and for any $t > 0$,

$$f(t) \le C_2(1 + t)^{-1}.$$

3. Proof of Theorem 1.1

In this section, we first establish the H^2 energy estimate and using a bootstrap argument to obtain Theorem 1.1.

Proof. **Step 1. L^2-energy norm**

Taking the L^2-inner product to (1.4) with (u_1, u_2, θ) respectively, we get, after a few calculations and integrations by parts,

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \eta\|\theta\|_{L^2}^2 \le 0. \tag{3.1}$$

Step 2. H^2-energy norm

Applying $\nabla\times$ to $(1.4)_1$, system (1.4) can be rewritten as

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \nu(\partial_{111}u_2 - \partial_{222}u_1) + \partial_1\theta, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \eta\theta = -u_2, \ (t,x)\in(0,\infty)\times\mathbb{R}^2, \end{cases} \tag{3.2}$$

here $\omega = \nabla\times u = \partial_1 u_2 - \partial_2 u_1$.

Applying ∇ to $(3.2)_1$, Δ to $(3.2)_2$, taking the L^2-inner product with $\nabla\omega$, $\Delta\theta$ respectively and integrating by parts then yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + \eta\|\Delta\theta\|_{L^2}^2 \\ &= -\langle\nabla(u\cdot\nabla\omega), \nabla\omega\rangle + \nu\langle\nabla(\partial_1^3 u_2 - \partial_2^3 u_1), \nabla\omega\rangle + \langle\nabla\partial_1\theta, \nabla\omega\rangle \\ &\quad - \langle\Delta(u\cdot\nabla\theta), \Delta\theta\rangle - \langle\Delta u_2, \Delta\theta\rangle \\ &= -\langle\nabla u\cdot\nabla\omega, \nabla\omega\rangle + \nu\langle\nabla(\partial_1^3 u_2 - \partial_2^3 u_1), \nabla\omega\rangle - \langle\Delta(u\cdot\nabla\theta), \Delta\theta\rangle \\ &= A_1 + A_2 + A_3, \end{aligned} \tag{3.3}$$

where we have used $\Delta u_2 = \partial_1\omega$.

Notice that $\omega = \partial_1 u_2 - \partial_2 u_1$. Making use of the Hölder inequality, the Sobolev inequality and the Young inequality, we obtain

$$\begin{aligned} A_1 &= -\int \nabla u\cdot\nabla\omega\cdot\nabla\omega \\ &\le C\|u\|_{H^2}\|\omega\|_{H^2}^2 \\ &\le C\|u\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned}$$

Easy computations based on integration by parts yield

$$\begin{aligned} A_2 &= \nu\int(\partial_1^3\nabla u_2 - \partial_2^3\nabla u_1)\cdot(\nabla\partial_1 u_2 - \nabla\partial_2 u_1) \\ &= \nu\int\partial_1^3\nabla u_2\cdot\nabla\partial_1 u_2 - \nu\int\partial_1^3\nabla u_2\cdot\nabla\partial_2 u_1 \\ &\quad - \nu\int\partial_2^3\nabla u_1\cdot\nabla\partial_1 u_2 + \nu\int\partial_2^3\nabla u_1\cdot\nabla\partial_2 u_1 \\ &= -\nu\int(\partial_1^2\nabla u_2)^2 - \nu\int\partial_1^3\nabla u_2\cdot\nabla\partial_2 u_1 \\ &\quad - \nu\int\partial_2^3\nabla u_1\cdot\nabla\partial_1 u_2 - \nu\int(\partial_2^2\nabla u_1)^2 \\ &= -\nu\int(|\partial_1\nabla^2 u_2|^2 + |\partial_2\nabla^2 u_1|^2). \end{aligned}$$

By applying Lemma 2.1, Lemma 2.2 and the Young inequality, we deduce

$$\begin{aligned}
A_3 &= -\int \Delta(u\cdot\nabla\theta)\cdot\Delta\theta\\
&= -\int \nabla^2 u\cdot\nabla\theta\cdot\Delta\theta - 2\int \nabla u\cdot\nabla^2\theta\cdot\Delta\theta\\
&= -\int \nabla^2 u_1\partial_1\theta\cdot\Delta\theta - \int \nabla^2 u_2\partial_2\theta\cdot\Delta\theta\\
&\quad - 2\int \nabla u_1\partial_1\nabla\theta\cdot\Delta\theta - 2\int \nabla u_2\partial_2\nabla\theta\cdot\Delta\theta\\
&\le C\|\Delta\theta\|_{L^2}\|\nabla^2 u_1\|_{L^2}^{\frac12}\|\partial_2\nabla^2 u_1\|_{L^2}^{\frac12}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1^2\theta\|_{L^2}^{\frac12}\\
&\quad + C\|\Delta\theta\|_{L^2}\|\nabla^2 u_2\|_{L^2}^{\frac12}\|\partial_1\nabla^2 u_2\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2^2\theta\|_{L^2}^{\frac12}\\
&\quad + C\|\nabla u_1\|_{L^\infty}\|\theta\|_{H^2}^2 + C\|\nabla u_2\|_{L^\infty}\|\theta\|_{H^2}^2\\
&\le C\|u\|_{H^2}^{\frac12}\|\partial_2 u_1\|_{H^2}^{\frac12}\|\theta\|_{H^2}^2 + C\|u\|_{H^2}^{\frac12}\|\partial_1 u_2\|_{H^2}^{\frac12}\|\theta\|_{H^2}^2\\
&\quad + C\|\nabla u_1\|_{H^1}^{\frac12}\|\partial_2\nabla u_1\|_{H^1}^{\frac12}\|\theta\|_{H^2}^2 + C\|\nabla u_2\|_{H^1}^{\frac12}\|\partial_1\nabla u_2\|_{H^1}^{\frac12}\|\theta\|_{H^2}^2\\
&\le C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\theta\|_{H^2}^2).
\end{aligned}$$

Collecting all the estimates above A_1 through A_3 leads to

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|u\|_{\dot H^2}^2+\|\theta\|_{\dot H^2}^2)+\eta\|\theta\|_{\dot H^2}^2+\nu\|\partial_1 u_2\|_{\dot H^2}^2+\nu\|\partial_2 u_1\|_{\dot H^2}^2\\
&\le C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\theta\|_{H^2}^2).
\end{aligned}\tag{3.4}$$

Combining (3.1) and (3.4), we have

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|u\|_{H^2}^2+\|\theta\|_{H^2}^2)+\eta\|\theta\|_{H^2}^2+\nu\|\partial_1 u_2\|_{H^2}^2+\nu\|\partial_2 u_1\|_{H^2}^2\\
&\le C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\theta\|_{H^2}^2).
\end{aligned}\tag{3.5}$$

Integrating over $[0,t]$ leads to, for some constant $C>0$,

$$\begin{aligned}
&\|u\|_{H^2}^2+\|\theta\|_{H^2}^2+2\int_0^t(\eta\|\theta\|_{H^2}^2+\nu\|\partial_1 u_2\|_{H^2}^2+\nu\|\partial_2 u_1\|_{H^2}^2)d\tau\\
&\le C(\|u_0\|_{H^2}^2+\|\theta_0\|_{H^2}^2)+C\sup_{0\le\tau\le t}(\|u\|_{H^2}+\|\theta\|_{H^2})\int_0^t(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\eta\|\theta\|_{H^2}^2)d\tau.
\end{aligned}$$

Step 3. Bootstrap argument

Now, we set

$$\mathcal{E} := \|u\|_{H^2}^2+\|\theta\|_{H^2}^2+2\int_0^t(\eta\|\theta\|_{H^2}^2+\nu\|\partial_1 u_2\|_{H^2}^2+\nu\|\partial_2 u_1\|_{H^2}^2)d\tau,$$

then we have

$$\mathcal{E}(t)\le \bar C_1\mathcal{E}(0)+\bar C_2\mathcal{E}^{\frac12}(t)\cdot\mathcal{E}(t)\le \bar C_1\mathcal{E}(0)+\bar C_2\mathcal{E}^{\frac32}(t).\tag{3.6}$$

We can infer that if $\|(u_0,\theta_0)\|_{H^2}$ is sufficiently small, then

$$\mathcal{E}(0)\le\frac{1}{16\bar C_1\bar C_2^2}\ \ or\ \ \|(u_0,\theta_0)\|_{H^2}\le\epsilon:=\frac{1}{4\bar C_2\sqrt{\bar C_1}},\tag{3.7}$$

therefore, the solution remains uniformly small, i.e.

$$\mathcal{E}(t) \leq C\epsilon^2.$$

The estimate (3.6) can thus obtain the desired stability result by making use of a bootstrap method. In fact, the method begins with the ansatz that, for $t \leq T$,

$$\mathcal{E}(t) \leq \frac{1}{4\bar{C}_2^2} := M.$$

(3.6) and (3.7) entail that

$$\mathcal{E}(t) \leq \bar{C}_1\mathcal{E}(0) + \bar{C}_2\mathcal{E}(t) \cdot \mathcal{E}(t)^{\frac{1}{2}} \leq \bar{C}_1\mathcal{E}(0) + \frac{1}{2}\mathcal{E}(t).$$

Then

$$\mathcal{E}(t) \leq 2\bar{C}_1\mathcal{E}(0) \leq \frac{1}{8\bar{C}_2^2} = \frac{M}{2}. \tag{3.8}$$

A simple bootstrap method implies that $T = \infty$. From (3.8), one has

$$\|u\|_{H^2}^2 + \|\theta\|_{H^2}^2 + 2\int_0^t (\eta\|\theta\|_{H^2}^2 + \nu\|\partial_1 u_2\|_{H^2}^2 + \nu\|\partial_2 u_1\|_{H^2}^2) d\tau \leq 2C_1\epsilon^2 \tag{3.9}$$

holds for any $t > 0$. This completes the proof of stability.

Step 4. Uniqueness

Assume that we are given $(u_1^{(1)}, u_2^{(1)}, \pi^{(1)}, \theta^{(1)})$ and $(u_1^{(2)}, u_2^{(2)}, \pi^{(2)}, \theta^{(2)})$ two solutions of (1.4) (with the same dates) satisfying the regularity assumptions of Theorem 1.1. In order to show that these two solutions coincide, we shall give estimate for $(\bar{u}_1, \bar{u}_2, \bar{\pi}, \bar{\theta}) := (u_1^{(1)} - u_1^{(2)}, u_2^{(1)} - u_2^{(2)}, \pi^{(1)} - \pi^{(2)}, \theta^{(1)} - \theta^{(2)})$ and $(\bar{u}_1, \bar{u}_2, \bar{\pi}, \bar{\theta})$ satisfy the following system:

$$\begin{cases} \partial_t \bar{u}_1 + (u^{(1)} \cdot \nabla)\bar{u}_1 + +(\bar{u} \cdot \nabla)u_1^{(2)} - \nu\partial_{22}\bar{u}_1 + \partial_1\bar{\pi} = 0, \\ \partial_t \bar{u}_2 + (u^{(1)} \cdot \nabla)\bar{u}_2 + +(\bar{u} \cdot \nabla)u_2^{(2)} - \nu\partial_{11}\bar{u}_2 + \partial_2\bar{\pi} = \bar{\theta}, \\ \partial_t \bar{\theta} + (u^{(1)} \cdot \nabla)\bar{\theta} + +(\bar{u} \cdot \nabla)\theta^{(2)} + \eta\bar{\theta} = -\bar{u}_2, \\ \nabla \cdot \bar{u} = 0, \bar{u}(x, 0) = \bar{\theta}(x, 0) = 0. \end{cases} \tag{3.10}$$

Taking the L^2-inner product of (3.10) with $(\bar{u}_1, \bar{u}_2, \bar{\theta})$, according to Lemma 2.1, the Young inequality and the uniformly global bounds for $\|(u_1^{(2)}, u_2^{(2)}, \theta^{(2)})\|_{H^2}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\bar{u}_1\|_{L^2}^2 + \|\bar{u}_2\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \eta\|\bar{\theta}\|_{L^2}^2 + \nu\|\partial_1\bar{u}_2\|_{L^2}^2 + \nu\|\partial_2\bar{u}_1\|_{L^2}^2 \\
&= -\int \bar{u}\cdot\nabla u_1^{(2)}\cdot\bar{u}_1 - \int \bar{u}\cdot\nabla u_2^{(2)}\cdot\bar{u}_2 - \int \bar{u}\cdot\nabla\theta^{(2)}\cdot\bar{\theta} \\
&= -\int \bar{u}\cdot\nabla u_1^{(2)}\cdot\bar{u}_1 - \int \bar{u}\cdot\nabla u_2^{(2)}\cdot\bar{u}_2 - \int \bar{u}_1\partial_1\theta^{(2)}\cdot\bar{\theta} - \int \bar{u}_2\partial_2\theta^{(2)}\cdot\bar{\theta} \\
&\le C\|\bar{u}\|_{L^2}\|\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\nabla u_1^{(2)}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_1^{(2)}\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\bar{u}\|_{L^2}\|\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\nabla u_2^{(2)}\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u_2^{(2)}\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\bar{\theta}\|_{L^2}\|\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta^{(2)}\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\theta^{(2)}\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\bar{\theta}\|_{L^2}\|\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta^{(2)}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\theta^{(2)}\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\bar{u}\|_{L^2}^{\frac{3}{2}}\|\partial_2\bar{u}_1\|_{L^2}^{\frac{1}{2}} + C\|\bar{u}\|_{L^2}^{\frac{3}{2}}\|\partial_1\bar{u}_2\|_{L^2}^{\frac{1}{2}} + C\|\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\bar{u}_1\|_{L^2}^{\frac{1}{2}}\|\bar{\theta}\|_{L^2} \\
&\quad + C\|\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\bar{u}_2\|_{L^2}^{\frac{1}{2}}\|\bar{\theta}\|_{L^2} \\
&\le \frac{\nu}{2}\|\partial_2\bar{u}_1\|_{L^2}^2 + \frac{\nu}{2}\|\partial_1\bar{u}_2\|_{L^2}^2 + C(\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2).
\end{aligned} \tag{3.11}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\bar{u}_1\|_{L^2}^2 + \|\bar{u}_2\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \eta\|\bar{\theta}\|_{L^2}^2 + \frac{\nu}{2}\|\partial_1\bar{u}_2\|_{L^2}^2 + \frac{\nu}{2}\|\partial_2\bar{u}_1\|_{L^2}^2 \\
&\le C(\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2),
\end{aligned} \tag{3.12}
$$

where $\bar{u} = (\bar{u}_1, \bar{u}_2)$. Grönwall inequality then implies

$$
\|\bar{u}\|_{L^2}^2 = \|\bar{\theta}\|_{L^2}^2 = 0. \tag{3.13}
$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, it gives the large-time behavior of the solution (u, θ) to (1.4).

Proof. From the inequality (3.9), one has $\int_0^\infty \|\theta\|_{H^2}^2 d\tau < \infty$. Taking the L^2-inner product of $(1.4)_3$ with θ, we have

$$
\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \eta\|\theta\|_{L^2}^2 = -\int u\cdot\nabla\theta\cdot\theta - \int u_2\theta = -\int u_2\theta. \tag{4.1}
$$

Applying Δ to $(1.4)_3$ and taking the L^2-inner product with $\Delta\theta$, we have

$$
\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^2}^2 + \eta\|\theta\|_{\dot{H}^2}^2 = -\int \Delta(u\cdot\nabla\theta)\cdot\Delta\theta - \int \Delta u_2\cdot\Delta\theta. \tag{4.2}
$$

Combining (4.1) and (4.2), by virtue of the Hölder inequality, Lemma 2.1, Lemma 2.2 and the Young inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^2}^2+\eta\|\theta\|_{H^2}^2\\
&=-\int u_2\theta-2\int\nabla u\cdot\nabla^2\theta\cdot\Delta\theta-\int\Delta u\cdot\nabla\theta\cdot\Delta\theta-\int\Delta u_2\cdot\Delta\theta\\
&=-\int u_2\theta-2\int\nabla u_1\partial_1\nabla\theta\cdot\Delta\theta-2\int\nabla u_2\partial_2\nabla\theta\cdot\Delta\theta\\
&\quad-\int\Delta u_1\partial_1\theta\cdot\Delta\theta-\int\Delta u_2\partial_2\theta\cdot\Delta\theta-\int\Delta u_2\cdot\Delta\theta\\
&\le C\|u_2\|_{H^2}\|\theta\|_{H^2}+C\|\nabla u_1\|_{H^1}^{\frac12}\|\partial_2\nabla u_1\|_{H^1}^{\frac12}\|\theta\|_{H^2}^2+C\|\nabla u_2\|_{H^1}^{\frac12}\|\partial_1\nabla u_2\|_{H^1}^{\frac12}\|\theta\|_{H^2}^2\\
&\quad+C\|\Delta\theta\|_{L^2}\|\Delta u_1\|_{L^2}^{\frac12}\|\partial_2\Delta u_1\|_{L^2}^{\frac12}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1^2\theta\|_{L^2}^{\frac12}\\
&\quad+C\|\Delta\theta\|_{L^2}\|\Delta u_2\|_{L^2}^{\frac12}\|\partial_1\Delta u_2\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2^2\theta\|_{L^2}^{\frac12}\\
&\le\frac{\eta}{2}\|\theta\|_{H^2}^2+C\|u_2\|_{H^2}^2+C\|u\|_{H^2}^2\|\theta\|_{H^2}^2\\
&\quad+C\|\theta\|_{H^2}^2\|\partial_1u_2\|_{H^2}^2+C\|\theta\|_{H^2}^2\|\partial_2u_1\|_{H^2}^2\\
&\le\frac{\eta}{2}\|\theta\|_{H^2}^2+C\|u\|_{H^2}^2(\|\theta\|_{H^2}^2+1)+C\|\theta\|_{H^2}^2(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2).
\end{aligned}\tag{4.3}
$$

We thus obtain

$$
\frac{d}{dt}\|\theta\|_{H^2}^2\le C\|u\|_{H^2}^2(\|\theta\|_{H^2}^2+1)+C\|\theta\|_{H^2}^2(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2).\tag{4.4}
$$

Multiplying both sides of (4.4) by $e^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}$, we end up with

$$
\frac{d}{dt}e^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}\|\theta\|_{H^2}^2\le Ce^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}\|u\|_{H^2}^2(\|\theta\|_{H^2}^2+1).\tag{4.5}
$$

Setting

$$
B(t)=e^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}\|\theta\|_{H^2}^2,
$$

then we get

$$
\frac{d}{dt}B(t)\le Ce^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}\|u\|_{H^2}^2(\|\theta\|_{H^2}^2+1).
$$

Integrating in time leads to, for any $0\le s<t$,

$$
B(t)-B(s)\le C\int_s^t e^{-C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}\|u\|_{H^2}^2(\|\theta\|_{H^2}^2+1)d\tau.
$$

Taking advantage of (3.9), we have

$$
B(t)-B(s)\le Ce^{-C\epsilon^2}\epsilon^2(\epsilon^2+1)(t-s),
$$

and we know $\int_0^\infty B(\tau)d\tau<\infty$, Lemma 2.3 provides

$$
B(t)\to 0\ as\ t\to\infty.
$$

From this, we deduce

$$
\|\theta(t)\|_{H^2}^2=e^{C\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2)d\tau}B(t)\to 0\ as\ t\to\infty.\tag{4.6}
$$

Next, we prove $\|(\omega(t), \nabla\theta(t))\|_{L^2} \leq C(1+t)^{\frac{1}{2}}$. Taking the L^2-inner product of $(3.2)_1$ with ω, applying ∇ to $(3.2)_2$ and taking the L^2-inner product of $(3.2)_2$ with $\nabla\theta$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta\|\nabla\theta\|_{L^2}^2\\ &= \nu\langle\partial_{111}u_2 - \partial_{222}u_1, \omega\rangle - \langle u\cdot\nabla\omega, \omega\rangle + \langle\partial_1\theta, \omega\rangle - \langle\nabla(u\cdot\nabla\theta), \nabla\theta\rangle - \langle\nabla u_2, \nabla\theta\rangle.\end{aligned}$$

Using integration by parts, $\nabla\cdot u = 0$, $\Delta u_2 = \partial_1\omega$ and $\omega = \partial_1 u_2 - \partial_2 u_1$ yields

$$\begin{aligned}&\langle u\cdot\nabla\omega, \omega\rangle = 0,\\ &\langle\partial_1\theta, \omega\rangle - \langle\nabla u_2, \nabla\theta\rangle = 0,\\ &\nu\langle\partial_{111}u_2 - \partial_{222}u_1, \omega\rangle = -\nu(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2).\end{aligned}$$

It follows from Lemma 2.1 and Theorem 1.1 that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta\|\nabla\theta\|_{L^2}^2 + \nu(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2)\\ &= -\int \nabla u_1\partial_1\theta\cdot\nabla\theta - \int\nabla u_2\partial_2\theta\cdot\nabla\theta\\ &\leq C\|\nabla\theta\|_{L^2}\|\nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\theta\|_{L^2}^{\frac{1}{2}}\\ &\quad + C\|\nabla\theta\|_{L^2}\|\nabla u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\theta\|_{L^2}^{\frac{1}{2}}\\ &\leq C(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\nabla\theta\|_{L^2}^2 + \|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2)\\ &\leq c\epsilon(\|\nabla\theta\|_{L^2}^2 + \|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2).\end{aligned} \tag{4.7}$$

Now, we take a sufficiency small constant ϵ in the above inequality, then terms on the right-hand side can be absorbed by the left-hand side, that is

$$\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + 2(\min\{\eta,\nu\} - c\epsilon)(\|\nabla\theta\|_{L^2}^2 + \|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2) \leq 0.$$

Integrating in time, for any $0 \leq s < t$, we have

$$\begin{aligned}\|\omega(t)\|_{L^2}^2 + &\|\nabla\theta(t)\|_{L^2}^2 + 2(\min\{\eta,\nu\} - c\epsilon)\int_s^t(\|\nabla\theta\|_{L^2}^2 + \|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2)d\tau\\ &\leq \|\omega(s)\|_{L^2}^2 + \|\nabla\theta(s)\|_{L^2}^2.\end{aligned}$$

Let $D(t) = \|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2$, we then deduce that $D(t)$ is a non-increasing function with respect to time. (3.9) implies again that

$$\int_0^\infty D(t)d\tau < \infty,$$

where used $\|\omega\|_{L^2}^2 \leq \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2$. Lemma 2.5 ensures

$$\|\omega\|_{L^2}, \|\nabla\theta\|_{L^2} \leq C(1+t)^{-\frac{1}{2}} \ as\ t \to 0.$$

Finally, we establish as $t \to 0$, $\|\partial_1\nabla u_2\|_{L^2} \to 0$, $\|\partial_2\nabla u_1\|_{L^2} \to 0$, $\|\partial_t u\|_{L^2} \to 0$.

After applying ∂_t to $(3.2)_1$ with $u_2 = -\partial_1(-\Delta)^{-1}\omega$, we have

$$\partial_{tt}\omega + \eta\partial_t\omega - \nu\partial_t(\partial_1^3 u_2 - \partial_2^3 u_1) - \mathcal{R}_1^2\omega = -\partial_1(u\cdot\nabla\theta) - \eta(u\cdot\nabla\omega) - \partial_t(u\cdot\nabla\omega) + \eta\nu(\partial_1^3 u_2 - \partial_2^3 u_1).$$

Taking the L^2-inner product of the above equality with $\partial_t\omega$, we get

$$\langle\partial_{tt}\omega, \partial_t\omega\rangle + \eta\langle\partial_t\omega, \partial_t\omega\rangle - \nu\langle\partial_t(\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t\omega\rangle - \langle\mathcal{R}_1^2\omega, \partial_t\omega\rangle - \eta\nu\langle(\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t\omega\rangle$$

$$= -\langle \partial_1(u\cdot\nabla\theta), \partial_t\omega\rangle - \eta\langle (u\cdot\nabla\omega), \partial_t\omega\rangle - \langle \partial_t(u\cdot\nabla\omega), \partial_t\omega\rangle.$$

A few calculations and integration by parts entail that

$$\begin{aligned}
-\nu\langle \partial_t(\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t\omega\rangle &= -\nu\langle \partial_t(\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t(\partial_1 u_2 - \partial_2 u_1)\rangle \\
&= -\nu\int \partial_t\partial_1^3 u_2\cdot\partial_t\partial_1 u_2 + \nu\int \partial_t\partial_2^3 u_1\cdot\partial_t\partial_1 u_2 \\
&\quad + \nu\int \partial_t\partial_1^3 u_2\cdot\partial_t\partial_2 u_1 - \nu\int \partial_t\partial_2^3 u_1\cdot\partial_t\partial_2 u_1 \\
&= \nu\int (\partial_t\partial_1^2 u_2)^2 + \nu\int (\partial_t\partial_2^2 u_1)^2 + \nu\int (\partial_t\partial_1\partial_2 u_2)^2 + \nu\int (\partial_t\partial_1\partial_2 u_1)^2 \\
&= \nu\int (|\partial_t\partial_1\nabla u_2|^2 + |\partial_t\partial_2\nabla u_1|^2) \\
&= \nu\|\partial_t\partial_1\nabla u_2\|_{L^2}^2 + \nu\|\partial_t\partial_2\nabla u_1\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
-\eta\nu\langle (\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t\omega\rangle &= -\eta\nu\langle (\partial_1^3 u_2 - \partial_2^3 u_1), \partial_t(\partial_1 u_2 - \partial_2 u_1)\rangle \\
&= -\eta\nu\int \partial_1^3 u_2\cdot\partial_t\partial_1 u_2 + \eta\nu\int \partial_1^3 u_2\cdot\partial_t\partial_2 u_1 \\
&\quad + \eta\nu\int \partial_2^3 u_1\cdot\partial_t\partial_1 u_2 - \eta\nu\int \partial_2^3 u_1\cdot\partial_t\partial_2 u_1 \\
&= \eta\nu\int \partial_1^2 u_2\partial_t\partial_1^2 u_2 + \eta\nu\int \partial_2^2 u_1\partial_t\partial_2^2 u_1 \\
&\quad + \eta\nu\int \partial_1\partial_2 u_2\partial_t\partial_1\partial_2 u_2 + \eta\nu\int \partial_1\partial_2 u_1\partial_t\partial_1\partial_2 u_1 \\
&= \eta\nu\int \partial_1\nabla u_2\cdot\partial_t\partial_1\nabla u_2 + \eta\nu\int \partial_2\nabla u_1\cdot\partial_t\partial_2\nabla u_1 \\
&= \eta\nu\frac{1}{2}\frac{d}{dt}(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t\omega\|_{L^2}^2 + \|\mathcal{R}_1\omega\|_{L^2}^2 + \eta\nu\|\partial_1\nabla u_2\|_{L^2}^2 + \eta\nu\|\partial_2\nabla u_1\|_{L^2}^2) \\
&\quad + \eta\|\partial_t\omega\|_{L^2}^2 + \nu\|\partial_t\partial_1\nabla u_2\|_{L^2}^2 + \nu\|\partial_t\partial_2\nabla u_1\|_{L^2}^2 \\
&= \int \partial_1(u\cdot\nabla\theta)\cdot\partial_t\omega - \eta\int u\cdot\nabla\omega\cdot\partial_t\omega - \int \partial_t(u\cdot\nabla\omega)\cdot\partial_t\omega \\
&= -\int \partial_1 u\cdot\nabla\theta\cdot\partial_t\omega - u\cdot\partial_1\nabla\theta\cdot\partial_t\omega - \eta\int u\cdot\nabla\omega\cdot\partial_t\omega - \int \partial_t u\cdot\nabla\omega\cdot\partial_t\omega \qquad (4.8)\\
&= -\int \partial_1 u\cdot\nabla\theta\cdot\partial_t\omega - \int u\cdot\partial_1\nabla\theta\cdot\partial_t\omega - \eta\int u\cdot\nabla\omega\cdot\partial_t\omega \\
&\quad - \int \partial_t u_1\partial_1\omega\cdot\partial_t\omega - \int \partial_t u_2\partial_2\omega\cdot\partial_t\omega \\
&= E_1 + E_2 + E_3 + E_4 + E_5.
\end{aligned}$$

For E_1, making use of the Hölder inequality and the Young inequality gives

$$\begin{aligned} E_1 &= -\int \partial_1 u \cdot \nabla\theta \cdot \partial_t\omega \\ &\le C\|\partial_1 u\|_{L^4}\|\nabla\theta\|_{L^4}\|\partial_t\omega\|_{L^2} \\ &\le C\|u\|_{H^2}\|\theta\|_{H^2}\|\partial_t\omega\|_{L^2} \\ &\le \frac{\eta}{10}\|\partial_t\omega\|_{L^2} + C\|u\|_{H^2}^2\|\theta\|_{H^2}^2. \end{aligned} \tag{4.9}$$

Similar as E_1, we have

$$\begin{aligned} E_2 &= -\int u \cdot \partial_1\nabla\theta \cdot \partial_t\omega \\ &\le C\|u\|_{L^\infty}\|\nabla\partial_1\theta\|_{L^2}\|\partial_t\omega\|_{L^2} \\ &\le C\|u\|_{H^2}\|\theta\|_{H^2}\|\partial_t\omega\|_{L^2} \\ &\le \frac{\eta}{10}\|\partial_t\omega\|_{L^2} + C\|u\|_{H^2}^2\|\theta\|_{H^2}^2, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} E_3 &= -\eta\int u \cdot \nabla\omega \cdot \partial_t\omega \\ &\le \eta\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\partial_t\omega\|_{L^2} \\ &\le C\|u\|_{H^2}\|\omega\|_{H^2}\|\partial_t\omega\|_{L^2} \\ &\le \frac{\eta}{10}\|\partial_t\omega\|_{L^2} + C\|u\|_{H^2}^2(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned} \tag{4.11}$$

In order to bound E_4, we exploit Lemma 2.1, the Hölder inequality and the Young inequality to get

$$\begin{aligned} E_4 &= -\int \partial_t u_1 \cdot \partial_1\omega \cdot \partial_t\omega \\ &\le C\|\partial_t u_1\|_{L^2}^{\frac12}\|\partial_2\partial_t u_1\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_1^2\omega\|_{L^2}^{\frac12}\|\partial_t\omega\|_{L^2} \\ &\le \frac{\eta}{20}\|\partial_t\omega\|_{L^2}^2 + C\|\partial_t u_1\|_{L^2}\|\partial_2\partial_t u_1\|_{L^2}\|\partial_1\omega\|_{L^2}\|\partial_1^2\omega\|_{L^2} \\ &\le \frac{\eta}{20}\|\partial_t\omega\|_{L^2}^2 + \frac{\eta}{20}\|\partial_t\partial_2 u_1\|_{L^2}^2 + C\|\partial_t u_1\|_{L^2}^2\|u\|_{H^2}^2(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2) \\ &\le \frac{\eta}{10}\|\partial_t\omega\|_{L^2}^2 + C\|\partial_t u_1\|_{L^2}^2\|u\|_{H^2}^2(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned} \tag{4.12}$$

Easy computations based on Lemma 2.1 also yield

$$E_5 \le \frac{\eta}{10}\|\partial_t\omega\|_{L^2}^2 + C\|\partial_t u_2\|_{L^2}^2\|u\|_{H^2}^2(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \tag{4.13}$$

Combining the last inequality with (4.8), (4.9), (4.10), (4.11) and (4.12), we have

$$\begin{aligned} &\frac{d}{dt}(\|\partial_t\omega\|_{L^2}^2 + \|\mathcal{R}_1\omega\|_{L^2}^2 + \eta\nu\|\partial_1\nabla u_2\|_{L^2}^2 + \eta\nu\|\partial_2\nabla u_1\|_{L^2}^2) \\ &\quad + \eta\|\partial_t\omega\|_{L^2}^2 + 2\nu\|\partial_t\partial_1\nabla u_2\|_{L^2}^2 + 2\nu\|\partial_t\partial_2\nabla u_1\|_{L^2}^2 \\ &\le C(\|u\|_{H^2}^2 + \|\partial_t u_1\|_{L^2}^2\|u\|_{H^2}^2 + \|\partial_t u_2\|_{L^2}^2\|u\|_{H^2}^2)(\|\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned} \tag{4.14}$$

We now prove $(\partial_t u_1, \partial_t u_2) \in L^\infty(0,\infty;L^2)$. $(1.4)_1$ can be rewritten as

$$\partial_t\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u\cdot\nabla u_1 \\ u\cdot\nabla u_2 \end{pmatrix} - \nu\begin{pmatrix} \partial_{22}u_1 \\ \partial_{11}u_2 \end{pmatrix} + \begin{pmatrix} \partial_1\pi \\ \partial_2\pi \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix}. \tag{4.15}$$

Recall that $\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$ stands for the Leray projector over divergence-free vector fields. Applying $\mathbb{P}$ to (4.15) and notice that

$$\begin{aligned}
\mathbb{P}\begin{pmatrix} u\cdot\nabla u_1 \\ u\cdot\nabla u_2\end{pmatrix} &= \begin{pmatrix} u\cdot\nabla u_1 - \partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u) \\ u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\end{pmatrix},\\
\mathbb{P}\nu\begin{pmatrix} \partial_{22}u_1 \\ \partial_{11}u_2\end{pmatrix} &= \nu\begin{pmatrix} \partial_{22}u_1 - \partial_1\Delta^{-1}\partial_1\partial_2^2u_1 - \partial_1\Delta^{-1}\partial_2\partial_1^2u_2 \\ \partial_{11}u_2 - \partial_2\Delta^{-1}\partial_1\partial_2^2u_1 - \partial_2\Delta^{-1}\partial_2\partial_1^2u_2\end{pmatrix}\\
&= \nu\begin{pmatrix} \partial_1\Delta^{-1}\partial_1\partial_1^2u_1 + \partial_2\Delta^{-1}\partial_2\partial_2^2u_1 \\ \partial_2\Delta^{-1}\partial_2\partial_2^2u_2 + \partial_1\Delta^{-1}\partial_1\partial_1^2u_2\end{pmatrix},\\
\mathbb{P}\begin{pmatrix} 0 \\ \theta\end{pmatrix} &= \begin{pmatrix} -\partial_1\Delta^{-1}\partial_2\theta \\ \theta - \partial_2\Delta^{-1}\partial_2\theta\end{pmatrix} = \begin{pmatrix} -\partial_1\Delta^{-1}\partial_2\theta \\ \partial_1\Delta^{-1}\partial_1\theta\end{pmatrix},
\end{aligned}$$

(4.15) is then converted into

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 - \partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u) - \nu\partial_2\Delta^{-1}\partial_2\partial_2^2u_1 - \nu\partial_1\Delta^{-1}\partial_1\partial_1^2u_1 = -\partial_1\Delta^{-1}\partial_2\theta, \\ \partial_t u_2 + u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u) - \nu\partial_2\Delta^{-1}\partial_2\partial_2^2u_2 - \nu\partial_1\Delta^{-1}\partial_1\partial_1^2u_2 = \partial_1\Delta^{-1}\partial_1\theta.\end{cases} \tag{4.16}$$

Using the fact that the singular integral operators $\mathcal{R}_{i,j} = \partial_i\partial_j(-\Delta)^{-1},\ i,j = 1,2$ are bounded on L^2, we have

$$\begin{aligned}
\|\partial_t u_1\|_{L^2} &\le \|u\cdot\nabla u_1\|_{L^2} + \|\partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^2}\\
&\quad + \|\partial_2\Delta^{-1}\partial_2\partial_2^2u_1\|_{L^2} + \|\partial_1\Delta^{-1}\partial_1\partial_1^2u_1\|_{L^2} + \|\partial_1\Delta^{-1}\partial_1\theta\|_{L^2}\\
&\le C\|u\|_{H^2}\|\omega\|_{L^2} + C\|\partial_2\nabla u_1\|_{L^2} + C\|\partial_1\nabla u_2\|_{L^2} + C\|\theta\|_{H^2}.
\end{aligned} \tag{4.17}$$

Similarly,

$$\|\partial_t u_2\|_{L^2} \le C\|u\|_{H^2}\|\omega\|_{L^2} + C\|\partial_2\nabla u_1\|_{L^2} + C\|\partial_1\nabla u_2\|_{L^2} + C\|\theta\|_{H^2}. \tag{4.18}$$

We then obtain $(\partial_t u_1, \partial_t u_2) \in L^\infty(0,\infty;L^2)$ by (3.9). Integrating in time to (4.14), we have

$$\begin{aligned}
&\|\partial_t\omega\|_{L^2}^2 + \|\mathcal{R}_1\omega\|_{L^2}^2 + \eta\nu\|\partial_1\nabla u_2\|_{L^2}^2 + \eta\nu\|\partial_2\nabla u_1\|_{L^2}^2 + \eta\int_0^t \|\partial_t\omega\|_{L^2}^2 d\tau\\
&\quad + 2\nu\int_0^t(\|\partial_t\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_t\partial_2\nabla u_1\|_{L^2}^2)d\tau\\
&\le \|\partial_t\omega_0\|_{L^2}^2 + \|\mathcal{R}_1\omega_0\|_{L^2}^2 + \eta\nu\|\partial_1\nabla u_{20}\|_{L^2}^2 + \eta\nu\|\partial_2\nabla u_{10}\|_{L^2}^2\\
&\quad + C\sup_{0\le\tau\le t}(\|u\|_{H^2}^2 + \|\partial_t u_1\|_{L^2}^2\|u\|_{H^2}^2 + \|\partial_t u_2\|_{L^2}^2\|u\|_{H^2}^2)\\
&\quad \times\int_0^t(\|\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2)d\tau,
\end{aligned}$$

which implies

$$\int_0^\infty \|\partial_t\omega\|_{L^2}^2 d\tau < \infty,\ \int_0^\infty(\|\partial_t\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_t\partial_2\nabla u_1\|_{L^2}^2)d\tau < \infty.$$

(3.9) shows that

$$\begin{aligned}
&\int_0^\infty \|\omega\|_{L^2}^2 d\tau < \int_0^\infty(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2)d\tau < \infty,\\
&\int_0^\infty(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_2\nabla u_1\|_{L^2}^2)d\tau < \int_0^\infty(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2)d\tau < \infty.
\end{aligned}$$

By Lemma 2.4, we obtain

$$\|\omega\|_{L^2}\to 0,\ \|\partial_1\nabla u_2\|_{L^2}\to 0,\ \|\partial_2\nabla u_1\|_{L^2}\to 0\ as\ t\to 0.$$

Of course, because when $t\to 0$, $\|\omega\|_{L^2}\to 0$, $\|\partial_1\nabla u_2\|_{L^2}\to 0$, $\|\partial_2\nabla u_1\|_{L^2}\to 0\ as\ t\to 0$, $\|\theta\|_{L^2}\to 0$, we can easily get from (4.17) and (4.18),

$$\|\partial_t u_1\|_{L^2}\to 0,\ \|\partial_t u_2\|_{L^2}\to 0,$$

namely

$$\|\partial_t u\|_{L^2}\to 0.$$

This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.5

In this section we show the proof of Theorem 1.5. To study the regularity and damping effects from the wave structure, we give the explicit representation.

Lemma 5.1. *Assume that g satisfies the degenerate wave type equation*

$$\begin{cases}\partial_{tt}g+(\eta+\nu\mathcal{R}_2^2\partial_2^2+\nu\mathcal{R}_1^2\partial_1^2)\partial_t g-(\mathcal{R}_1^2-\nu\eta\mathcal{R}_2^2\partial_2^2-\nu\eta\mathcal{R}_1^2\partial_1^2)g=0,\\ g(x,0)=g_0(x),\partial_t g(x,0)=g_1(x).\end{cases}\tag{5.1}$$

Then g can be explicitly represented as

$$g=G_1(g_1+\frac{1}{2}(\eta+\nu\mathcal{R}_2^2\partial_2^2+\nu\mathcal{R}_1^2\partial_1^2)g_0)+G_2 g_0,\tag{5.2}$$

where G_1 and G_2 are defined as in (1.10)*, namely*

$$\widehat{G_1}(\xi,t)=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1},\ \widehat{G_2}(\xi,t)=\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t}).\tag{5.3}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2+(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})\lambda+(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})=0$$

or

$$\lambda_1=-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})-\frac{1}{2}\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})},\tag{5.4}$$

$$\lambda_2=-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})+\frac{1}{2}\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4(\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2})}.\tag{5.5}$$

When $\lambda_1=\lambda_2$, (5.2) *remains valid if we replace $\widehat{G_1}$ and $\widehat{G_2}$ in* (5.3) *by their corresponding limit form, namely*

$$\widehat{G_1}=\lim_{\lambda_2\to\lambda_1}\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1},\ \widehat{G_2}=\lim_{\lambda_2\to\lambda_1}\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t})=e^{\lambda^1 t}.$$

Proof. The details of this proof are similar to Lemma 3.1 in [10], so we omit it. □

Next we provide precise upper bounds on the Fourier multiplier operator G_1 and G_2.

Lemma 5.2. *Suppose S_1, S_2, S_3 and A denote the following subsets of $\mathbb{R}^2$,*

$$S_1 := \{\xi \in \mathbb{R}^2 : \frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2} \geq \frac{3}{16}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2\},$$

$$S_2 := \{\xi \in \mathbb{R}^2 : \frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2} < \frac{3}{16}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 \text{ and } \xi \in A^c\},$$

$$S_3 := \{\xi \in \mathbb{R}^2 : \frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2} < \frac{3}{16}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 \text{ and } \xi \in A\},$$

$$A := \{\xi \in \mathbb{R}^2 : \xi_1^2 \leq \frac{2\eta\rho(t)}{\nu\eta - 2\nu\eta\rho(t)},\ \xi_2^2 \leq \frac{2\eta\rho(t)}{\nu\eta - 2\nu\eta\rho(t)}\},$$

where A^c is the complement of A and $0 < \rho(t) < \min\{\frac{\eta}{2}, \frac{1}{2\eta}\}$ is specified in (5.11). Then $\widehat{G_1}(\xi,t)$ and $\widehat{G_2}(\xi,t)$ have the following upper bounds.

(i) For any $\xi \in S_1$,

$$Re\lambda_1 \leq -\frac{1}{2}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),\ Re\lambda_2 \leq -\frac{1}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),$$

$$|\widehat{G_1}(\xi,t)| \leq te^{-\frac{1}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t},\ |\widehat{G_2}(\xi,t)| \leq Ce^{-\frac{1}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}.$$

(ii) For any $\xi \in S_2$,

$$\lambda_1 \leq -\frac{3}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),\ \lambda_2 \leq -\rho(t),$$

$$|\widehat{G_1}(\xi,t)| \leq \frac{2}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}(e^{-\frac{3}{4}\eta t} + e^{-\rho(t)t}),\ |\widehat{G_2}(\xi,t)| \leq C(e^{-\frac{3}{4}\eta t} + e^{-\rho(t)t}).$$

(iii) For any $\xi \in S_3$,

$$\lambda_1 \leq -\frac{3}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),\ \lambda_2 \leq 0,$$

$$|\widehat{G_1}(\xi,t)| \leq \frac{2}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}(e^{-\frac{3}{4}\eta t} + 1),\ |\widehat{G_2}(\xi,t)| \leq C(e^{-\frac{3}{4}\eta t} + 1).$$

Proof. (i) For any $\xi \in S_1$, we split S_1 into two parts:

$$S_{11} := \{\xi \in S_1 : (\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 \geq 4\frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}\},$$

$$S_{12} := \{\xi \in S_1 : S_1/S_{11}\}.$$

When $\xi \in S_{11}$, we have

$$0 \leq (\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 - 4\frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2} \leq \frac{1}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2.$$

Meanwhile, $\lambda_1,\ \lambda_2$ are real and

$$\lambda_1 \leq -\frac{1}{2}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),\ \lambda_2 \leq -\frac{1}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}).$$

With the help of the mean-value theorem, one then has

$$|\widehat{G_1}(\xi,t)| = |\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}| \leq te^{-\frac{1}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}.$$

For $\widehat{G_2}$, we can directly obtain

$$|\widehat{G_2}(\xi,t)| \leq \frac{1}{2}|e^{\lambda_1 t}+e^{\lambda_2 t}| \leq Ce^{-\frac{1}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}.$$

When $\xi \in S_{12}$, we have

$$(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 < 4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}.$$

Furthermore, both λ_1 and λ_2 are a pair of complex conjugates, namely

$$\lambda_1 = -\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 - \frac{i}{2}\sqrt{4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}-(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2},$$

$$\lambda_2 = -\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 + \frac{i}{2}\sqrt{4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}-(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2}.$$

Thanks to Euler's formula and the important limits, we obtain

$$\begin{aligned}|\widehat{G_1}(\xi,t)| &\leq e^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}\left|\frac{\sin(t\sqrt{4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}-(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2})}{\sqrt{4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}-(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2}}\right| \\ &\leq te^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t},\end{aligned}$$

and

$$|\widehat{G_2}(\xi,t)| \leq e^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}.$$

(ii) For any $\xi \in S_2$, we have

$$(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 - 4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2} \geq \frac{1}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 > 0.$$

At the same time, λ_1 and λ_2 are real and

$$\lambda_1 \leq -\frac{3}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}).$$

We bound λ_2 as follows:

$$\begin{aligned}\lambda_2 &= -\frac{1}{2}\left(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}-\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}\right) \\ &= -2\frac{\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}{\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}+\sqrt{(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2-4\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}} \\ &\leq -\frac{\frac{\xi_1^2}{|\xi|^2}+\nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}{\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}} = -\frac{\xi_1^2+\nu\eta(\xi_1^4+\xi_2^4)}{\eta(\xi_1^2+\xi_2^2)+\nu(\xi_1^4+\xi_2^4)}.\end{aligned}$$

Setting

$$S_{21}=\{\xi\in S_2, \xi_1^2 \geq \xi_2^2\},\ S_{22}=\{\xi\in S_2, \xi_1^2 < \xi_2^2\}.$$

When $\xi \in S_{21}$,

$$\lambda_2 \leq -\frac{\nu\eta\xi_1^4}{2\eta\xi_1^2 + 2\nu\xi_1^4} \leq -\frac{\nu\eta\xi_1^2}{2\eta + 2\nu\xi_1^2},$$

when $\xi \in S_{22}$,

$$\lambda_2 \leq -\frac{\nu\eta\xi_2^4}{2\eta\xi_2^2 + 2\nu\xi_2^4} \leq -\frac{\nu\eta\xi_2^2}{2\eta + 2\nu\xi_2^2}.$$

Since $\xi \in A^c$, we have $\lambda_2 \leq -\rho(t)$. Then

$$\begin{aligned}|\widehat{G_1}| &\leq \frac{1}{\sqrt{(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})^2 - 4\frac{\xi_1^2}{|\xi|^2} + \nu\eta\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}}(e^{-\frac{3}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t} + e^{-\rho(t)t}) \\ &\leq \frac{2}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}(e^{-\frac{3}{4}\eta t} + e^{-\rho(t)t}),\end{aligned}$$

and

$$\begin{aligned}|\widehat{G_2}| &\leq C(e^{-\frac{3}{4}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t} + e^{-\rho(t)t}) \\ &\leq C(e^{-\frac{3}{4}\eta t} + e^{-\rho(t)t}).\end{aligned}$$

(iii) For any $\xi \in S_3$, according to (5.4) and (5.5), we easily get

$$\lambda_1 \leq \frac{3}{4}(\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}),\ \lambda_2 \leq 0,$$

from which we obtain

$$|\widehat{G_1}| \leq \frac{2}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}(e^{-\frac{3}{4}\eta t} + 1),\ |\widehat{G_2}| \leq (e^{-\frac{3}{4}\eta t} + 1).$$

This completes the proof of Lemma 5.2. □

Now we pay attention to prove Theorem 1.5 by Lemma 5.1 and Lemma 5.2.

Proof. Combining Lemma 5.1 and (1.9), we have

$$u_1 = G_1(\partial_t u_1)(x,0) + \frac{1}{2}(\eta + \nu\mathcal{R}_2^2\partial_2^2 + \nu\mathcal{R}_1^2\partial_1^2)G_1 u_{10} + G_2 u_{10}, \tag{5.6}$$

$$u_2 = G_1(\partial_t u_2)(x,0) + \frac{1}{2}(\eta + \nu\mathcal{R}_2^2\partial_2^2 + \nu\mathcal{R}_1^2\partial_1^2)G_1 u_{20} + G_2 u_{20}, \tag{5.7}$$

$$\theta = G_1(\partial_t \theta)(x,0) + \frac{1}{2}(\eta + \nu\mathcal{R}_2^2\partial_2^2 + \nu\mathcal{R}_1^2\partial_1^2)G_1 \theta_0 + G_2 \theta_0. \tag{5.8}$$

Letting $t = 0$ in the following system:

$$\begin{cases} \partial_t u_1 - \nu\partial_2\Delta^{-1}\partial_2\partial_2^2 u_1 - \nu\partial_1\Delta^{-1}\partial_1\partial_1^2 u_1 + \partial_1\Delta^{-1}\partial_2\theta = 0, \\ \partial_t u_2 - \nu\partial_2\Delta^{-1}\partial_2\partial_2^2 u_2 - \nu\partial_1\Delta^{-1}\partial_1\partial_1^2 u_2 - \partial_1\Delta^{-1}\partial_1\theta = 0, \\ \partial_t\theta + \eta\theta + u_2 = 0, \end{cases} \tag{5.9}$$

From (5.9), (5.7) and (5.8) we have

$$u_1 = \frac{1}{2}(\eta - \nu\mathcal{R}_2^2\partial_2^2 - \nu\mathcal{R}_1^2\partial_1^2)G_1 u_{10} - \partial_1\Delta^{-1}\partial_2\theta + G_2 u_{10},$$

$$u_2 = \frac{1}{2}(\eta - \nu\mathcal{R}_2^2\partial_2^2 - \nu\mathcal{R}_1^2\partial_1^2)G_1u_{20} + \partial_1\Delta^{-1}\partial_1\theta + G_2u_{20},$$
$$\theta = -\frac{1}{2}(\eta - \nu\mathcal{R}_2^2\partial_2^2 - \nu\mathcal{R}_1^2\partial_1^2)G_1\theta_0 - G_1u_{20} + G_2\theta_0.$$

Using Plancherel's Theorem entails

$$\begin{aligned}\|u_1\|_{L^2}^2 &\le C\int |\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2|\widehat{G_1}|^2|\widehat{u_{10}}|^2d\xi + \int |\frac{\xi_1\xi_2}{|\xi|^2}|^2|\widehat{G_1}|^2|\widehat{\theta_0}|^2d\xi\\ &\quad + \int |\widehat{G_2}|^2|\widehat{u_{10}}|^2d\xi\\ &= I_1 + I_2 + I_3.\end{aligned}\tag{5.10}$$

For the term I_1, from Lemma 5.2 and the inequality $(a^2x)^{b^2}e^{-a^2x} \le C$ for any $a, b \in \mathbb{R}$, $x \ge 0$ we have

$$\begin{aligned}I_1 &= \int_{S_1} |\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2|\widehat{G_1}|^2|\widehat{u_{10}}|^2d\xi + \int_{S_2} |\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2|\widehat{G_1}|^2|\widehat{u_{10}}|^2d\xi\\ &\quad + \int_{S_3} |\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2|\widehat{G_1}|^2|\widehat{u_{10}}|^2d\xi\\ &\le C\int |\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2t^2e^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}|\widehat{u_{10}}|^2d\xi + C\int \left|\frac{\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}\right|^2(e^{-\frac{3}{2}\eta t} + e^{-2\rho(t)t})|\widehat{u_{10}}|^2d\xi\\ &\quad + C\int_A \left|\frac{\eta - \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}\right|^2(e^{-\frac{3}{2}\eta t} + 1)|\widehat{u_{10}}|^2d\xi\\ &\le C\int |\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}|^2t^2e^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}|\widehat{u_{10}}|^2d\xi + C\int (e^{-\frac{3}{2}\eta t} + e^{-2\rho(t)t})|\widehat{u_{10}}|^2d\xi\\ &\quad + C\int_A |\widehat{u_{10}}|^2d\xi\\ &\le C(e^{-C\eta t} + e^{-2\rho(t)t})\|u_{10}\|_{L^2}^2 + C\frac{2\eta\rho(t)}{\nu\eta - 2\nu\rho(t)}\|u_{10}\|_{L^1}^2.\end{aligned}$$

For I_2, Lemma 5.2 entails that

$$\begin{aligned}I_2 &\le C\int_{S_1} t^2e^{-\frac{1}{2}(\eta+\nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2})t}|\widehat{\theta_0}|^2d\xi + C\int \left|\frac{1}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}\right|^2(e^{-\frac{3}{2}\eta t} + e^{-2\rho(t)t})|\widehat{\theta_0}|^2d\xi\\ &\quad + C\int_{S_3} \left|\frac{1}{\eta + \nu\frac{\xi_1^4+\xi_2^4}{|\xi|^2}}\right|^2(e^{-\frac{3}{2}\eta t} + 1)|\widehat{\theta_0}|^2d\xi\\ &\le C\int e^{-C\eta t}|\widehat{\theta_0}|^2d\xi + C\int (e^{-\frac{3}{2}\eta t} + e^{-2\rho(t)t})|\widehat{\theta_0}|^2d\xi\\ &\quad + C\int_A |\widehat{\theta_0}|^2d\xi\\ &\le C(e^{-C\eta t} + e^{-2\rho(t)t})\|\theta_0\|_{L^2}^2 + C\frac{2\eta\rho(t)}{\nu\eta - 2\nu\rho(t)}\|\theta_0\|_{L^1}^2.\end{aligned}$$

Similar as I_2, the bound of I_3 easily follows that

$$I_3 \leq C(e^{-C\eta t} + e^{-2\rho(t)t})\|u_{10}\|_{L^2}^2 + C\frac{2\eta\rho(t)}{\nu\eta - 2\nu\rho(t)}\|u_{10}\|_{L^1}^2.$$

Combining the estimates of I_1, I_2 and I_3, we have

$$\|u_1\|_{L^2}^2 \leq C(e^{-C\eta t} + e^{-2\rho(t)t})(\|u_{10}\|_{L^2}^2 + \|\theta_0\|_{L^2}^2) + C\frac{2\eta\rho(t)}{\nu\eta - 2\nu\rho(t)}(\|u_{10}\|_{L^1}^2 + \|\theta_0\|_{L^2}^2).$$

Without loss of generality, we can assume that $t \geq 1$. In particular, from our assumption, if we choose

$$\rho(t) = \min\{\frac{\eta}{2}, \frac{1}{2\eta}\}(1+t)^{-\sigma} \; for \; any \; t \geq 1, \; 0 < \sigma < 1 \; . \tag{5.11}$$

then we have the upper bounds

$$e^{-2\rho(t)t} = e^{-\frac{2\min\{\frac{\eta}{2}, \frac{1}{2\eta}\}t}{(1+t)^{-\sigma}}} \leq e^{-\min\{\frac{\eta}{2}, \frac{1}{2\eta}\}(1+t)^{1-\sigma}} \leq C(1+t)^{-\alpha}, \; \forall \alpha > 0$$

and

$$\frac{2\eta\rho(t)}{\nu\eta - 2\nu\rho(t)} = \frac{2\eta\rho(t)}{\nu\eta(1 - \frac{2\nu\rho(t)}{\eta})} \leq \frac{\eta(1+t)^{-\sigma}}{\nu(1 - (1+t)^{-\sigma})} \leq C(\sigma)(1+t)^{-\sigma}.$$

As a consequence we have the following estimate for u_1:

$$\|u_1(t)\|_{L^2}^2 \leq C(\sigma)(1+t)^{-\sigma}\|(u_0, \theta_0)\|_{L^1 \cap L^2}.$$

In a similar manners, we can obtain the estimates of $\|u_{20}\|_{L^2}^2$ and $\|\theta_0\|_{L^2}^2$ and they are omitted for simplicity. This complete the proof of Theorem 1.5. □

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