# Quantum Tsallis-Jensen-Shannon divergence and its bounds

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March 07, 2024

**Abstract**

In this paper, we first introduce some properties of quantum Tsallis-Jensen-Shannon divergence for two density matrices. Then one of the beautiful and very simple applications of Hermit-Hadamard's inequality [S. Dragomir, et. al. Math. Methods Appl. Sci., 1–15 (2020)] for log-convexity in the concepts of quantum entropies and quantum Tsallis-Jensen-Shannon is given.

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**Keywords:** *Hermit-Hadamard's inequality;* log-*convexity; Density matrices;Tsallis quantum relative entropy; Quantum relative entropy; quantum Tsallis-Jensen-Shannon divergence.*

## 1 Introduction

An important tool in the study of discrepancy between two quantum density matrices is the class of Quantum divergences. This class of divergences has many applications in quantum information theory [32, 35, 6, 17, 19, 13, 26]. A famous and important quantum divergence is the von Numaann entropy.

**Definition 1** *The quantum version of the Shannon entropy for the density matrix $\rho$ is called von Neumann entropy and is given by [33, 32]*

$$S\left(\rho\right)=\mathrm{Tr}\left[-\rho\log\rho\right]. \tag{1}$$

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This quantum divergence which plays an important role in quantifying and discrimination of entanglement, is the quantum version of the classical Kullback-Leibler divergence [12].

**Definition 2** *For two density matrices $\rho$ and $\delta$, von Neumann relative entropy (or quantum relative entropy) of $\rho$ with respect to $\delta$ is given by [31, 27, 32]*

$$S(\rho|\delta) = \begin{cases} \operatorname{Tr}\left[\rho\left(\log\rho - \log\delta\right)\right] & if\ \ supp\ \rho \leq\ \ supp\ \delta \\ +\infty & otherwise, \end{cases} \tag{2}$$

**Remark 3** *The quantum relative entropy (2) is a quantum version of Kullback and Leibler divergence [12].*

Similar to its classical version, the von Numaann relative entropy isn't symmetric and does not obey the triangle inequality [28, 32]. The symmetric version of the von Numaann entropy is the quantum Jensen-Shannon divergence. This symmetric quantum divergence has been studied in may physical systems [13, 20, 14, 7, 26].

**Definition 4** *The quantum Jensen-Shannon divergence of $\rho$ with respect to $\delta$ is given by*

$$J(\rho|\delta) = \frac{1}{2}\left[S(\rho|\frac{\rho+\delta}{2}) + S(\delta|\frac{\rho+\delta}{2})\right] = S(\frac{\rho+\delta}{2}) - \frac{1}{2}S(\rho) - \frac{1}{2}S(\delta).$$

*Here $S(\rho|\delta)$ is the von Neumann relative entropy (2) and $S(\rho)$ is the von Neumann entropy (1).*

A well-known generalization of Shannon entropy, is Tsallis entropy [29], which is a useful to for the study of statistical systems with long range interaction [30, 10].

In 2003, Abe [1] introduced a quantum version of Tsallis relative entropy.

**Definition 5** *For two density matrices $\rho$ and $\delta$, one-parametric Tsallis quantum relative entropy of $\rho$ with respect to $\delta$ is given by*

$$D_\alpha(\rho|\delta) = \frac{1 - \operatorname{Tr}\left[\rho^{1-\alpha}\delta^{\alpha}\right]}{\alpha} = -\ln_\alpha\left[\left(\operatorname{Tr}\left[\rho^{1-\alpha}\delta^{\alpha}\right]\right)^{\frac{1}{\alpha}}\right],\ 0 < \alpha \leqslant 1, \tag{3}$$

*which*

$$\ln_\alpha(\delta) = \frac{\delta^\alpha - 1}{\alpha},$$

*is the Tsallis logarithm [29, 8].*

More details on Tsallis entropy can be found in [30, 10, 2, 3].

**Definition 6** *[5] For two density matrices $\rho$ and $\delta$,*
*(I) the Tsallis-Lin quantum relative entropy of $\rho$ to $\delta$ is given by*

$$D_\alpha^{Lin}(\rho|\delta) = D_\alpha(\rho|\frac{\rho+\delta}{2}) = \frac{1 - \mathrm{Tr}\left[2^{-\alpha}\rho^{1-\alpha}(\rho+\delta)^\alpha\right]}{\alpha}, \ 0 < \alpha \leqslant 1.$$

*(II) the quantum Tsallis-Jensen-Shannon divergence of $\rho$ to $\delta$ is given by*

$$\begin{aligned} J_\alpha(\rho|\delta) &= \frac{1}{2}\left[D_\alpha(\rho|\frac{\rho+\delta}{2}) + D_\alpha(\delta|\frac{\rho+\delta}{2})\right] \\ &= \frac{1}{2}\left[D_\alpha^{Lin}(\rho|\delta) + D_\alpha^{Lin}(\delta|\rho)\right], \end{aligned} \tag{4}$$

*for any $0 < \alpha \leqslant 1$.*

**Theorem 7** *[5] For two density matrices $\rho$ and $\delta$, we have*

$$\hat{L}D_\alpha(\rho|\delta) \leqslant D_\alpha(\rho|\delta) \leqslant \hat{R}D_\alpha(\rho|\delta),$$

*where,*

$$\hat{L}D_\alpha(\rho|\delta) = (1 - a_0)\left(\frac{a_0 + 1}{2}\right)^{\alpha-1} \tag{5}$$

*and*

$$\hat{R}D_\alpha(\rho|\delta) = (1 - a_0)\,\frac{a_0^{\alpha-1} + 1}{2}. \tag{6}$$

*Here* $a_0 = \left(\mathrm{Tr}\left[\rho^{1-\alpha}\delta^\alpha\right]\right)^{\frac{1}{\alpha}}, 0 < \alpha \leqslant 1.$

Notice that as an extension of one-parametric Tsallis quantum relative entropy $D_\alpha(\rho|\delta)$ in Definition 5, two-parametric Tsallis quantum relative entropy are introduced as follows

$$D_{\alpha,\beta}(\rho|\delta) = \frac{1 - \left(\mathrm{Tr}\left[\rho^{1-\beta}\delta^\beta\right]\right)^{\frac{\alpha}{\beta}}}{\alpha} = -\ln_\alpha\left[\left(\mathrm{Tr}\left[\rho^{1-\beta}\delta^\beta\right]\right)^{\frac{1}{\beta}}\right], \tag{7}$$

for any $\alpha, \beta \in (0, 1]$. Clearly, if $\alpha = \beta$, then $D_{\alpha,\beta}(\rho|\delta) = D_{\alpha}(\rho|\delta)$.

Some basic mathematical concepts, despite their simplicity, still surprise us with new applications. One of the beauties of mathematics on convex function is Hermit-Hadamard's inequality [9, 23, 24]. This inequality was first introduced by Ch. Hermit. Then J. Hadamard proved and completed it again ten years later in 1883 [22, 23]. This inequality states that if $f$ is a convex function on $[a_0, a_1]$, then

$$f\left(\frac{a_0 + a_1}{2}\right) \leq \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} f(z)dz \leq \frac{f(a_0) + f(a_1)}{2}.$$

The concept of log-convexity is a stronger property of convexity. A positive function on $[a_0, a_1]$ is log-convex if $\log f(z)$ is a convex function of $z$. If $f$ is a log-convex function on $[a_0, a_1]$, then [24]

$$\begin{aligned} f\left(\frac{a_0 + a_1}{2}\right) \leq \exp\left(\frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \log f(z)dz\right) &\leq \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} f(z)dz \\ \leq M\left\{f(a_0), f(a_1)\right\} &\leq \frac{f(a_0) + f(a_1)}{2}, \end{aligned} \tag{8}$$

where

$$M\left\{r, s\right\} = \begin{cases} \frac{r-s}{\log r - \log s} & r \neq s, \\ r & r = s, \end{cases} \tag{9}$$

which is called the logarithmic mean.

The paper is organized as follows. In next section, some properties of two parametric Tsallis quantum relative entropy $D_{\alpha,\beta}(\rho|\delta)$, two parametric Tsallis-Lin quantum relative entropy $D^{Lin}_{\alpha,\beta}(\rho|\delta)$ and two parametric quantum Jensen-Shannon divergence $J_{\alpha,\beta}(\rho|\delta)$ are discussed. Finally, some conclusions are given.

## 2 Main results

**Theorem 8** *For two density matrices $\rho$ and $\delta$, we have*

$$(1 - a_0) \exp\left\{\frac{1-\alpha}{1-a_0}(1 - a_0 + a_0 \log a_0)\right\} \leqslant D_{\alpha,\beta}(\rho|\delta) \leqslant (1 - a_0)\, M\left\{a_0^{\alpha-1}, 1\right\},$$

*where* $a_0 = \left(\mathrm{Tr}\left[\rho^{1-\beta}\delta^{\beta}\right]\right)^{\frac{1}{\beta}}$ *for any* $\alpha, \beta \in (0,1]$ *and* $D_{\alpha,\beta}(\rho|\delta)$ *and* $M\{\cdot,\cdot\}$ *are defined in (7), (9), respectively.*

**Proof.** If we suppose $f(z) = z^{\alpha-1}, 0 < \alpha \leqslant 1$ and $0 < z \leqslant 1$, then

$$\frac{\partial^2 (\log f(z))}{\partial z^2} = \frac{1-\alpha}{z^2} \geqslant 0,$$

for any $0 < \alpha \leqslant 1$. So, the function $f(z)$ is log-convex on $(0,1]$. Clearly, for two density matrices $\rho$ and $\delta$, Hölder's inequality [4, 18] implies that

$$\mathrm{Tr}\left[\rho^{1-\beta}\delta^{\beta}\right] \leqslant (\mathrm{Tr}[\rho])^{1-\beta}(\mathrm{Tr}[\delta])^{\beta} = 1.$$

So,

$$0 \leqslant a_0 = \left(\mathrm{Tr}\left[\rho^{1-\beta}\delta^{\beta}\right]\right)^{\frac{1}{\beta}} \leqslant 1,$$

for any $0 < \beta \leqslant 1$. Now, by Hermite-Hadamard's inequality for log-convex function (8), we have

$$\begin{aligned}
(1-a_0) f\left(\frac{a_0+1}{2}\right) &= (1-a_0)\left(\frac{a_0+1}{2}\right)^{\alpha-1} \\
&\leq (1-a_0)\exp\left(\frac{1}{1-a_0}\int_{a_0}^{1}\log f(z)dz\right) \\
&= (1-a_0)\exp\left\{\frac{1-\alpha}{1-a_0}(1-a_0+a_0\log a_0)\right\} \\
&\leq \int_{a_0}^{1} f(z)dz = \frac{1-a_0^{\alpha}}{\alpha} \\
&\leq (1-a_0) M\left\{a_0^{\alpha-1}, 1\right\} \\
&\leq \frac{(1-a_0)\left(a_0^{\alpha-1}+1\right)}{2}.
\end{aligned} \tag{10}$$

This completes the proof. □

In special case, if $\alpha = \beta$, then we have new bounds which are better than the bounds of Theorem 8 obtained by Asgharzadeh Jelodar *et al.* [5] in 2021.

**Corollary 9** *For two density matrices $\rho$ and $\delta$, we have*

$$\hat{L}D_\alpha(\rho|\delta) \leqslant LD_\alpha(\rho|\delta) \leqslant D_\alpha(\rho|\delta) \leqslant RD_\alpha(\rho|\delta) \leqslant \hat{R}D_\alpha(\rho|\delta)$$

*where*

$$\begin{aligned} LD_\alpha(\rho|\delta) &= (1-a_0)\exp\left\{\frac{1-\alpha}{1-a_0}\left(1-a_0+a_0\log a_0\right)\right\}, \\ RD_\alpha(\rho|\delta) &= (1-a_0)\, M\left\{a_0^{\alpha-1}, 1\right\}, \end{aligned}$$

$a_0 = \left(\mathrm{Tr}\left[\rho^{1-\alpha}\delta^\alpha\right]\right)^{\frac{1}{\alpha}}, 0 < \alpha \leqslant 1,$ *and* $\hat{L}D_\alpha(\rho|\delta), \hat{R}D_\alpha(\rho|\delta), D_\alpha(\rho|\delta)$ *and* $M\{\cdot,\cdot\}$ *are defined in (5), (6), (3) and (9), respectively.*

**Example 10** *Let $\sigma(q) = \frac{1-q}{4}I_{4\times4} + q|\Phi^+\rangle\langle\Phi^+|$ be bipartite mixed Werner state [34], where $0 \le q \le 1$, and $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$ is the maximally entangled Bell state. By using the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, this density matrix can be written as follows [21]*

$$\sigma(q) = \frac{1}{4}\begin{pmatrix} 1+q & 0 & 0 & 2q \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 1-q & 0 \\ 2q & 0 & 0 & 1+q \end{pmatrix}.$$

*It is easy to show that $\sigma(q)$ is an entangled density matrix for $\frac{1}{3} < q \le 1$ and is separable density for $0 \le q \le \frac{1}{3}$ [21].*

*Let us to fix $\rho = \sigma(q = 0.1)$ and $\delta = \sigma(q = 0.9)$ here. Figure 1 presents the plots of $D_{\alpha,\beta}(\rho|\delta)$ for any $0 < \alpha \leqslant 1$ with different values of $\beta$. Figure 2 presents the plot of $D_\alpha(\rho|\delta)$ with its upper bound $RD_\alpha(\rho|\delta)$ and its lower bound $LD_\alpha(\rho|\delta)$, which are given in Corollary 9 . Here $\hat{R}D_\alpha(\rho|\delta)$ and $\hat{L}D_\alpha(\rho|\delta)$ are respectively the proposed upper and lower bounds of $D_\alpha(\rho|\delta)$ in Theorem 8 obtained by Asgharzadeh Jelodar et al. [5]*

**Definition 11** *For two density matrices $\rho$ and $\delta$,*

*(I) two parametric Tsallis-Lin quantum relative entropy of $\rho$ to $\delta$ is given by*

$$D^{Lin}_{\alpha,\beta}(\rho|\delta) = D_{\alpha,\beta}(\rho|\frac{\rho+\delta}{2}) = \frac{1-\left(\mathrm{Tr}\left[\rho^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^\beta\right]\right)^{\frac{\alpha}{\beta}}}{\alpha},$$

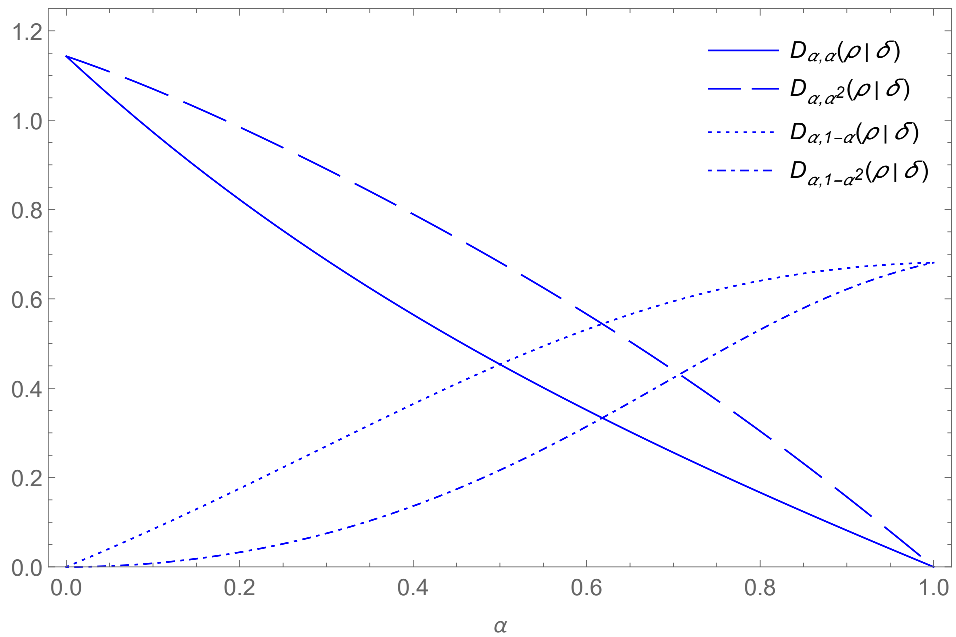


Figure 1: Plots of $D_{\alpha,\beta}(\rho|\delta)$ for any $0 < \alpha \leqslant 1$ with some different $\beta$. Here two density matrices $\rho$ and $\delta$ are given in Example 10.

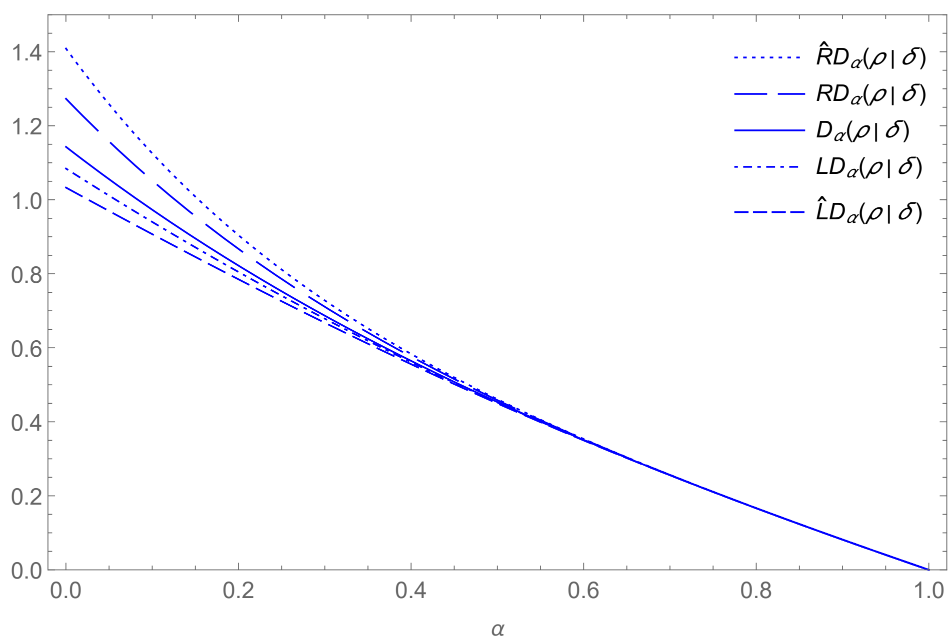


Figure 2: Plots of $D_{\alpha}(\rho|\delta)$ and their bounds proposed in Corollary 9 . Here two density matrices $\rho$ and $\delta$ are given in Example 10.

*for any* $\alpha, \beta \in (0, 1]$.

*(II) two parametric quantum Tsallis-Jensen-Shannon divergence of* $\rho$ *to* $\delta$ *is given by*

$$\begin{aligned} J_{\alpha,\beta}(\rho|\delta) &= \frac{1}{2}\left[D_{\alpha,\beta}(\rho|\frac{\rho+\delta}{2}) + D_{\alpha,\beta}(\delta|\frac{\rho+\delta}{2})\right] \\ &= \frac{1}{2}\left[D^{Lin}_{\alpha,\beta}(\rho|\delta) + D^{Lin}_{\alpha,\beta}(\delta|\rho)\right] \\ &= \frac{1}{2\alpha}\left[2 - \left(\mathrm{Tr}\left[\rho^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^{\beta}\right]\right)^{\frac{\alpha}{\beta}} - \left(\mathrm{Tr}\left[\delta^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^{\beta}\right]\right)^{\frac{\alpha}{\beta}}\right], \end{aligned} \tag{11}$$

*for any* $\alpha, \beta \in (0, 1]$.

**Example 12** *Using* $\rho = \sigma(q = 0.1)$ *and* $\delta = \sigma(q = 0.9)$*, where* $\sigma(q)$ *is given by Example 10, we plotted Tsallis-Jensen-Shannon divergence* $J_{\alpha,\beta}(\rho|\delta)$ *for any* $0 < \alpha < 1$ *with some different* $\beta$ *in Figure 3.*

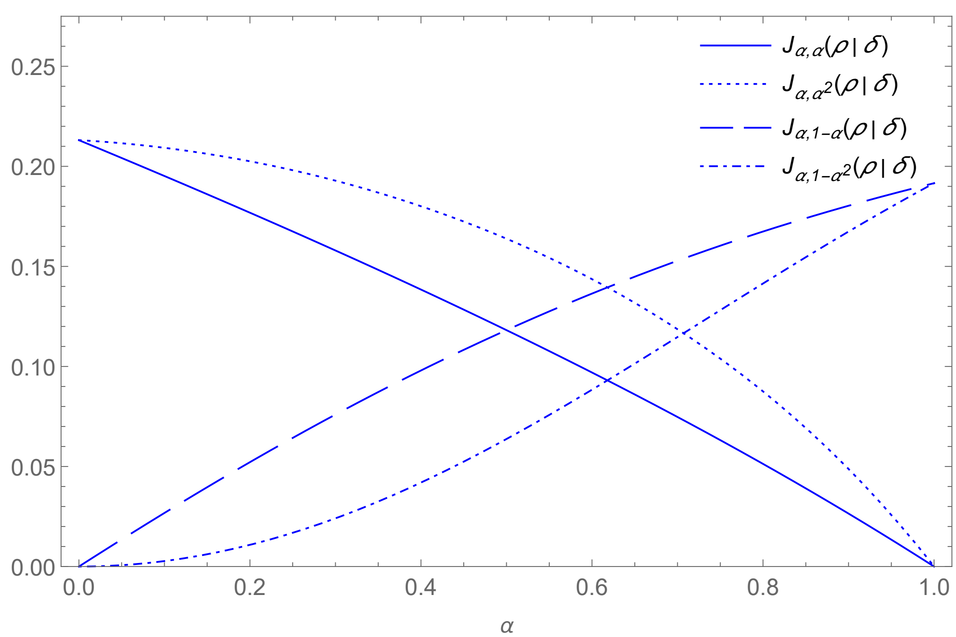


Figure 3: Plots of two parametric quantum Tsallis-Jensen-Shannon divergence $J_{\alpha,\beta}(\rho|\delta)$ for any $0 < \alpha < 1$ with some different $\beta$. Two density matrices $\rho$ and $\delta$ are given in Example 10.

**Corollary 13** *For two density matrices $\rho$ and $\delta$, the lower and upper bounds of quantum Tsallis-Lin relative entropy $D^{Lin}_{\alpha,\beta}(\rho|\delta)$ are given by*

$$(1-b_0)\exp\left\{\frac{\alpha-1}{b_0-1}\left(1+b_0\log\left(b_0\right)-b_0\right)\right\}\leqslant D^{Lin}_{\alpha,\beta}(\rho|\delta)\leqslant(1-b_0)\,M\left\{b_0^{\alpha-1},1\right\},$$

*where $b_0=\left(\mathrm{Tr}\left[\rho^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^{\beta}\right]\right)^{\frac{1}{\beta}}$ and $\alpha,\beta\in(0,1]$.*

**Corollary 14** *The lower and upper bounds of two parametric quantum Tsallis-Jensen-Shannon divergence $J_{\alpha,\beta}(\rho|\delta)$ are given by*

$$\begin{aligned}LJ_{\alpha,\beta}(\rho|\delta)&=\frac{1}{2}\left[(1-b_0)\,e^{\frac{\alpha-1}{b_0-1}(1+b_0\log(b_0)-b_0)}+(1-b_1)\,e^{\frac{\alpha-1}{b_1-1}(1+b_1\log(b_1)-b_1)}\right]\\&\leqslant J_{\alpha,\beta}(\rho|\delta)\\&\leqslant\frac{1}{2}\left[(1-b_0)\,M\left\{b_0^{\alpha-1},1\right\}+(1-b_1)\,M\left\{b_1^{\alpha-1},1\right\}\right]=RJ_{\alpha,\beta}(\rho|\delta),\end{aligned}$$

*where $b_0=\left(\mathrm{Tr}\left[\rho^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^{\beta}\right]\right)^{\frac{1}{\beta}}$ and $b_1=\left(\mathrm{Tr}\left[\delta^{1-\beta}\left(\frac{\rho+\delta}{2}\right)^{\beta}\right]\right)^{\frac{1}{\beta}}$ for $\alpha,\beta\in(0,1]$.*

**Example 15** *Using $\rho=\sigma(q=0.1)$ and $\delta=\sigma(q=0.9)$, where $\sigma(q)$ is given by Example 10, we plotted the differences of quantum Tsallis-Jensen-Shannon divergence and its upper bound $RJ_{\alpha,\alpha}(\rho|\delta)-J_{\alpha,\alpha}(\rho|\delta)$ (dotted curve), and its difference with its proposed lower bound $J_{\alpha,\alpha}(\rho|\delta)-LJ_{\alpha,\alpha}(\rho|\delta)$ , $\alpha\in(0,1]$ (dashed curve) in Figure 4.*

## 3 Conclusion

In this paper, we have discussed the nonadditive extensions of quantum divergences, i.e., a two parametric version of the non-extensive a quantum Tsallis-von Neumann entropy, Tsallis-Lin quantum relative entropy and quantum Tsallis-Jensen-Shannon divergence of two density matrices. Some properties of them have been proposed. Using the Hölder's inequality and Hermite-Hadamard's inequality for the class of log-convex functions, we proposed some bounds for the two parametric version Tsallis quantum relative entropy $D_{\alpha,\beta}(\rho|\delta)$ in Theorem 8. The proposed bounds are stronger than the former bounds in the special case of $\alpha=\beta$ (see

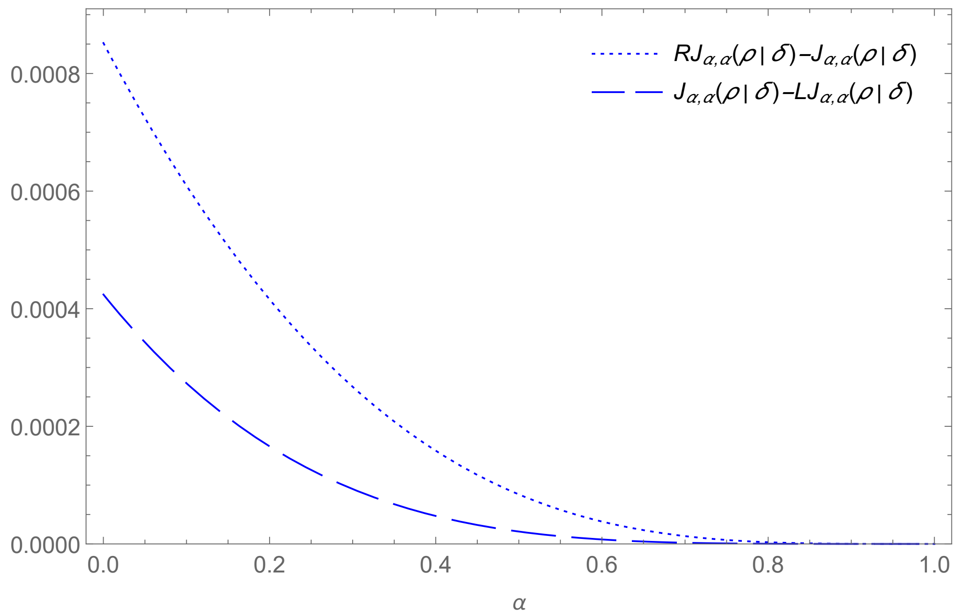


Figure 4: Plots of $RJ_{\alpha,\alpha}(\rho|\delta) - J_{\alpha,\alpha}(\rho|\delta)$, and $J_{\alpha,\alpha}(\rho|\delta) - LJ_{\alpha,\alpha}(\rho|\delta)$ as functions of to the $\alpha \in (0, 1]$. $\rho$ and $\delta$ are same as the given density matrices in Example 15.

Corollary 9). We also propose a two parametric Tsallis-Lin quantum relative entropy and of $\rho$ and $\delta$, $D^{Lin}_{\alpha,\beta}(\rho|\delta)$ and find some strong bounds for it in Corollary 13. We also introduce a generalized two parametric quantum Jensen-Shannon divergence $J_{\alpha,\beta}(\rho|\delta)$ and find some new bounds for it in Corollary 14. Our results can be applicable in quantum information theory of systems with long-range interactions.

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