An existence result for implicit functional equations

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In this article, we attempt to provide a more general method based on Petryshyn's fixed-point theorem to ensure the existence of solutions to implicit functional equations. These implicit functional equations include fractional, non-fractional, (fractional) stochastic integral equations, etc., and any combination of them in $C(\ I)$. Some results regarding the existence of fixed points in implicit functional integral equations will be reviewed in the literature. We show that this general result unifies and improves many of the main results in the literature. To illustrate that our approach is more general than other methods, we present some concrete examples. Also, we apply our method to create new functional equations in practice and check the existence of solutions.

An existence result for implicit functional equations

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Abstract

In this article, we attempt to provide a more general method based on Petryshyn's fixed-point theorem to ensure the existence of solutions to implicit functional equations. These implicit functional equations include fractional, non-fractional, (fractional) stochastic integral equations, etc., and any combination of them in $C(I)$. Some results regarding the existence of fixed points in implicit functional integral equations will be reviewed in the literature. We show that this general result unifies and improves many of the main results in the literature. To illustrate that our approach is more general than other methods, we present some concrete examples. Also, we apply our method to create new functional equations in practice and check the existence of solutions.

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1 Introduction and preliminaries

There are many results on the existence of one- or two-dimensional dimensional nonlinear integral equations through measures of noncompactness (for instance, see some of them in the references). The motivation of this article is to unify and expand them in a single and simple way. We will use a general scheme that shows that many results can be embedded in it and can be useful and practical for researchers interested in this subject. Throughout this paper, assume $(\mathfrak{B}, ||\cdot||)$ be a Banach space. Denote $B_\rho = \{x \in \mathfrak{B} : ||x|| \leq \rho\}$ for a closed ball of radius $\rho > 0$ centered at 0, ∂B_ρ for the boundary of B_ρ, $B_\rho(E) = \{x \in E : ||x||_u \leq \rho\}$, and

$$\mathfrak{B}_\rho(E) = \{\Psi : B_\rho(E) \to E, \Psi \text{ is a continuous functional}\}.$$

Suppose $E = C(I) = C(I, \mathbb{R})$ be a Banach algebra of continuous functions $f : I \to \mathbb{R}$ with ordinary point-wise summation and multiplication and the uniform norm $||x||_u = \sup\{|x(s)|, s \in I\}$, $I := [a_1, b_1] \times \cdots \times [a_r, b_r] \subset \mathbb{R}^r$ with the Euclidean metric $|\cdot|$ (as a particular case $I := [a, b] \subset \mathbb{R}$), and Ω set of continuous and non-decreasing functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0, \forall t > 0, 0 < \phi(t) < t$,
In this article, we intend to investigate the fixed point existence solution of functional equation $Tz = z, z \in B_\rho(E)$ in general, $T : B_\rho(E) \to E$ is defined as

$$Tz(s) = \zeta\left(s, \Psi_1(z)(s), \ldots, \Psi_n(z)(s), \Phi_1(z)(s), \ldots, \Phi_m(z)(s)\right), \qquad s \in I, z \in B_\rho(E). \tag{1}$$

where $\zeta, \Psi_i, \Phi_j \in \mathfrak{B}_\rho(E), i = 1, \ldots, n, j = 1, \ldots, m$ are completely defined in Theorem 2.1.
The functional equation $Tz = z$ is general in the sense that it includes many forms of well-known integral equations considered in the articles, see Section 2.1.

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The quantity

$$\omega(z,\sigma) = \sup\{|z(s) - z(\bar{s})| : s, \bar{s} \in I, |s - \bar{s}| \leq \sigma\}.$$

is called the modulus of continuity of $z \in E$. Also, for all bounded sets $S \subset E$ the quantity

$$\chi(S) = \lim_{\sigma \to 0} \omega_{\sup}(S,\sigma), \tag{2}$$

defines a measure of noncompactness (briefly, MN) on E [6], where

$$\omega_{\sup}(S,\sigma) = \sup\{\omega(z,\sigma), z \in S\}.$$

An MN, in general, can be defined on a Banach space $(\mathfrak{B}, ||\cdot||)$. Properties about it may be found in the books of fixed point theory, for instance, some good books on the subject include [2, 6, 7, 10, 30, 32]. It is well known that if α is an MN in a Banach space $\mathfrak{B}$ then

(i) $\alpha(B) = 0$ iff B is a precompact set in $\mathfrak{B}$,

(ii) $\alpha(\lambda B) = |\lambda|\alpha(B)$, where $\lambda B = \{\lambda z : z \in B\}$,

(iii) $\alpha(A + B) \leq \alpha(A) + \alpha(B), A, B \subseteq \mathfrak{B}$.

Definition 1.1 ([2, 30])**.** 1. Let $T : \mathfrak{B} \to \mathfrak{B}$ be a map, and α be an MN on $\mathfrak{B}$. Then T is called a completely continuous compact map if T is continuous and T maps bounded sets to precompact sets. Let $\rho > 0$. Denote

$$\mathfrak{B}^{\mathrm{C}}_{\rho}(E) = \{T \in \mathfrak{B}_{\rho}(E), \alpha(T(S)) = 0, \forall S \subset B_{\rho}(E)\}.$$

2. $T \in \mathfrak{B}_{\rho}(E)$ is called a condensing map if

$$\alpha(TS) < \alpha(S), \quad \forall\ S \subseteq B_{\rho}, \alpha(S) > 0.$$

3. $T \in \mathfrak{B}_{\rho}(E)$ is called a k-set contraction $(0 \leq k)$ if

$$\alpha(TS) \leq k\alpha(S), \quad \forall\ S \subseteq B_{\rho}, \alpha(S) > 0.$$

From properties (i)-(ii), every $\Phi \in \mathfrak{B}^{\mathrm{C}}_{\rho}(E)$ is a completely continuous compact map. Let us denote $\mathfrak{L}_{\rho}(E) \subseteq \mathfrak{B}_{\rho}(E)$ for the set of 1-set contraction. Every functional $\Psi \in \mathfrak{B}_{\rho}(E)$ such that

$$\exists \sigma > 0, \forall z \in B_{\rho}(E), \omega(\Psi(z),\sigma) \leq \omega(z,\sigma),$$

as well as every non-expansive or Lipschitz functional with Lipschitz constant 1 are in $\mathfrak{L}_{\rho}(E)$ (see [29, Example 2.2.14] and [16]), also, examples of $\Psi \in \mathfrak{L}_{\rho}(E)$ are in Subsection 2.1.
In this paper, the tool for investigating the existence solution of implicit functional equation (1) is a variant of Darbo and Schauder fixed point theorem from Petryshyn [28].

Theorem 1.2 (see also [29, 30])**.** Let $T : B_{\rho} \to \mathfrak{B}$ be a continuous and condensing map such that

$$\text{if} \quad T(z) = kz, \quad \text{for some } z \in \partial B_{\rho}, \quad \text{then } k \leq 1,$$

or

$$T(\partial B_{\rho}) \subseteq B_{\rho}.$$

Then T has at least one fixed point in B_{ρ}.

2 Main results

Theorem 2.1. (I) Let T be defined by Eq. (1), where $\Psi_i, \Phi_j \in \mathfrak{B}_\rho^{\mathrm{C}}(E), j = 1, \dots, m, i = 1, \dots, n, \rho > 0$, then T is a continuous functional.

(II) Let $\rho > 0$ be a real number such that $\Phi_j \in \mathfrak{B}_\rho^{\mathrm{C}}(E), j = 1, \dots, m, \Psi_i \in \mathfrak{L}_\rho(E), i = 1, \dots, n$ and $\zeta \in C(I \times \prod_{i=1}^n [-M_i, M_i] \times \prod_{i=1}^m [-N_i, N_i], \mathbb{R})$, where

$$M_i = \sup\{|\Psi_i(z)(s)| : s \in I, z \in B_\rho(E)\}, N_j = \sup\{|\Phi_j(z)(s)| : s \in I, z \in B_\rho(E)\},$$

and there exist non-negative constants $k_j, j = 1, \cdots, m$ and, $\phi_i \in \Omega, i = 1, \dots, n$ such that $\phi = \sum_{i=1}^n \phi_i \in \Omega$ and

$$|\zeta(s, u_1, \dots, u_n, v_1, \dots, v_m) - \zeta(s, \bar{u}_1, \dots, \bar{u}_n, \bar{v}_1, \dots, \bar{v}_m)| \leq \sum_{i=1}^n \phi_i(|u_i - \bar{u}_i|) + \sum_{i=1}^m k_i |v_i - \bar{v}_i|, \quad (3)$$

for all $s \in I, u_1, \bar{u}_1, \in [-M_1, M_1], \dots, u_n, \bar{u}_n \in [-M_n, M_n], v_1, \bar{v}_1, \in [-N_1, N_1], \dots, v_m, \bar{v}_m \in [-N_m, N_m]$. Then T is a condensing map.

(III) Furthermore, if ζ satisfies

$$\sup\{|\zeta(s, u_1, \dots, u_n, v_1, \dots, v_m)| : s \in I, u_i \in [-M_i, M_i], v_j \in [-N_j, N_j]\} \leq \rho, \quad (4)$$

then T has at least one fixed point in $B_\rho(E)$.

Proof. We prove the theorem only for $n = m = 1$, assertion for any $n, m \in \mathbb{N}$ holds by induction (or a similar way). It is clear that the functional T is a combination of continuous functionals Ψ_i, Φ_j from $B_\rho(E)$ into $C(I)$, so it is well defined.

(I) Choose $\varepsilon > 0$ then by continuity of Ψ_1, Φ_1 and since ϕ is non-decreasing and continuous function there exists $\delta > 0$ such that for any $||y - x||_u < \delta$ we get $\phi(||\Psi_1(x) - \Psi_1(y)||_u) < \frac{\varepsilon}{2}$ and $||\Phi_1(x) - \Phi_1(y)||_u < \frac{\varepsilon}{2k_1}$, thus from (3) we have

$$\begin{aligned} |(Tx)(s) - (Ty)(s)| &= |\zeta\left(s, \Psi_1(x)(s), \Phi_1(x)(s)\right) - \zeta\left(s, \Psi_1(y)(s), \Phi_1(y)(s)\right)| \\ &\leq \phi(|\Psi_1(x)(s) - \Psi_1(y)(s)|) + k_1 \left|\Phi_1(x)(s) - \Phi_1(y)(s)\right|, \end{aligned} \quad (5)$$

and

$$||Tx - Ty||_u \leq \phi(||\Psi_1(x) - \Psi_1(y)||_u) + k_1 ||\Phi_1(x) - \Phi_1(y)||_u \leq \varepsilon.$$

This shows that T is a continuous functional.

(II) Let $\sigma > 0$, $z \in S$, where S is a bounded subset of $B_\rho(E)$, $\chi(S) > 0$ and $s_1, s_2 \in I$ with $|s_2 - s_1| \leq \sigma$.

Case 1: If there exists term $\Psi_1 \in \mathfrak{L}_\rho(E)$ in Eq. (1) such that (3) holds then we have

$$\begin{aligned} |(Tz)(s_2) - (Tz)(s_1)| &= |\zeta\left(s_2, \Psi_1(z)(s_2), \Phi_1(z)(s_2)\right) - \zeta\left(s_1, \Psi_1(z)(s_1), \Phi_1(z)(s_1)\right)| \\ &\leq |\zeta\left(s_2, \Psi_1(z)(s_2), \Phi_1(z)(s_2)\right) - \zeta\left(s_2, \Psi_1(z)(s_2), \Phi_1(z)(s_1)\right)| \\ &+ |\zeta\left(s_2, \Psi_1(z)(s_2), \Phi_1(z)(s_1)\right) - \zeta\left(s_2, \Psi_1(z)(s_1), \Phi_1(z)(s_1)\right)| \\ &+ |\zeta\left(s_2, \Psi_1(z)(s_1), \Phi_1(z)(s_1)\right) - \zeta\left(s_1, \Psi_1(z)(s_1), \Phi_1(z)(s_1)\right)| \\ &\leq \phi(|\Psi_1(z)(s_2) - \Psi_1(z)(s_1)|) + k_1 |\Phi_1(z)(s_2) - \Phi_1(z)(s_1)| + \omega^1(\zeta, \sigma) \\ &\leq \phi(\omega(\Psi_1(z), \sigma)) + k_1 \omega(\Phi_1(z), \sigma) + \omega^1(\zeta, \sigma), \end{aligned}$$

where

$$\omega^1(\zeta,\sigma) = \sup\{|\zeta(s,u_1,v_1) - \zeta(\bar{s},u_1,v_1)| : |s-\bar{s}| \le \sigma, s,\bar{s} \in I, u_1 \in [-M_1,M_1], v_1 \in [-N_1,N_1]\}.$$

Thus, we get

$$\omega_{\sup}(T(S),\sigma) \le \phi(\omega_{\sup}(\Psi_1(S),\sigma)) + k_1\omega_{\sup}(\Phi_1(S),\sigma) + \omega^1(\zeta,\sigma). \tag{6}$$

From the above relations and assumptions $\Psi_1 \in \mathfrak{L}_\rho(E)$, $\Phi_1 \in \mathfrak{B}^{\mathrm{C}}_\rho(E)$ and continuity of ζ and ϕ, taking limit as $\sigma \to 0$, we get

$$\chi(T(S)) \le \phi(\chi(S)) < \chi(S). \tag{7}$$

Thus, T is a condensing map.

Case 2: If there exist no terms $\Psi_1 \in \mathfrak{L}_\rho(E)$ in Eq. (1) then by a similar method as above instead of inequality (6) we have

$$\omega_{\sup}(T(S),\sigma) \le k_1\omega_{\sup}(\Phi_1(S),\sigma) + \omega^1(\zeta,\sigma).$$

Taking limit as $\sigma \to 0$, we get $\chi(T(S)) = 0$, thus, $\chi(T(S)) < \phi(\chi(S))$ holds and T is a condensing map.

(III) Let $z \in \partial B_\rho(E)$ and $Tz = kz$ then we have $||Tz||_u = k||z||_u = k\rho$ and by assumptions (III) we get

$$||Tz||_u = \sup_{s\in I}|Tz(s)| = \sup_{s\in I}|\zeta\left(s,\Psi_1(z)(s),\Phi_2(z)(s)\right)| \le \rho,$$

hence, $||Tz||_u \le \rho$, thus, $k||z||_u = k\rho = ||Tz||_u \le \rho$, i.e. $k \le 1$, thus, the result follows from Theorem 1.2.

□

Remark 2.2. 1. If condition (III) does not hold then Eq. (1) may not have a solution, for instance, consider the following Fredholm integral equation which has no solution in E (see [26, Subsection 11.2])

$$T(z)(s) = z(s) = s + \Phi(z)(s),$$
$$\Phi(z)(s) = \int_0^1 k(s,t)z(t)dt, k(s,t) = \begin{cases} \pi^2 t(1-s), & t \le s \\ \pi^2 s(1-t), & s \le t, \end{cases}$$

where $s \in [0,1]$, $z \in E$. Here for all $\rho > 0$ we have $\Phi \in \mathfrak{B}^{\mathrm{C}}_\rho(E)$ (see Example 2.3-(1) below) and in Eq. (1) we have $\zeta(s,v) = s + v$.

2. Assume that there exist non-negative constants $k_j, j = 1,\cdots,m$, $\phi_i \in \Omega, i = 1,\ldots,n$ such that $\phi = \sum_{i=1}^n \phi_i \in \Omega$ and for all $s \in I$ then $\zeta \in C(I\times\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R})$ defined as

$$\zeta(s,u_1,\ldots,u_n,v_1,\ldots,v_m) = \sum_{i=1}^n \phi_i(u_i) + \sum_{j=1}^m k_j v_j,$$

can be an example in condition (II) of Theorem 2.1. Also, let $l_i, i = 1,\cdots,n$ be non-negative constants such that $\sum_{i=1}^n l_i < 1$ then $\phi_i(t) = l_i t$ satisfies condition (II) of Theorem 2.1.

3. With the same assumptions in Theorem 2.1, let $M_\zeta = \sup\{|\zeta(s,0,\dots,0)| : s \in I\}$ and assume that there exists $\rho > 0$ such that
$$\sum_{i=1}^{n} \phi_i(M_i) + \sum_{i=1}^{m} k_i N_i + M_\zeta \le \rho, \tag{8}$$
then conditions (II) and (8) imply condition (III), since
$$\begin{aligned}
&\sup\{|\zeta(s,u_1,\dots,u_n,v_1,\dots,v_m)| : s \in I, u_i \in [-M_i, M_i], v_j \in [-N_j, N_j]\}\\
&\quad\le \sup\{|\zeta(s,u_1,\dots,u_n,v_1,\dots,v_m) - \zeta(s,0,\dots,0)| : s \in I, u_i \in [-M_i, M_i], v_j \in [-N_j, N_j]\}\\
&\quad+ \sup\{|\zeta(s,0,\dots,0)| : s \in I\}\\
&\quad\le \sum_{i=1}^{n} \phi_i(M_i) + \sum_{i=1}^{m} k_i N_i + M_\zeta \le \rho.
\end{aligned}$$

4. From Theorem 1.2, it is clear condition (III) can be replaced by
$$\exists \rho > 0; \sup\{|\zeta(s,u_1,\dots,u_n,v_1,\dots,v_m)| : s \in I, u_i \in [-M_i^\rho, M_i^\rho], v_j \in [-N_j^\rho, N_j^\rho]\} \le \rho,$$
where
$$M_i^\partial = \sup\{|\Psi_i(z)(s)| : s \in I, z \in \partial B_\rho(E)\}, N_j^\partial = \sup\{|\Phi_j(z)(s)| : s \in I, z \in \partial B_\rho(E)\}.$$

5. Note that condition (III) implies that $T(B_\rho(E)) \subseteq B_\rho(E)$ and if there is no term $\Phi \in \mathfrak{B}_\rho^{\mathrm{C}}(E)$ in (1) and $\Psi_i \in \mathfrak{L}_\rho(E), i = 1,\dots,n$ are Lipschitz functional with Lipschitz constant 1 then the conclusion of Theorem 2.1 follows from Boyd and Wond's theorem [9] too, this situation is considered in [8, Theorem 2] for product of two maps.

Example 2.3. Let $I = [a,b]$, $r_0 > 0$, $\theta \in C(I), D = \sup\{\theta(s), s \in I\}, \eta \in C([0,D]), \eta([0,D]) \subseteq I$ and $K \in C(I \times [0,D] \times [0,r_0], \mathbb{R})$.

(1) Let $\Phi \in \mathfrak{B}_{r_0}(E)$ be defined as follows
$$\Phi(z)(s) = \int_0^{\theta(s)} K(s,\xi,z(\eta(\xi)))d\xi, z \in B_{r_0}(E).$$
Assume that $|s_2 - s_1| \le \sigma$ then we have
$$\begin{aligned}
|\Phi(z)(s_2) - \Phi(z)(s_1)| &= \left|\int_a^{\theta(s_2)} K(s_2,\xi,z(\eta(\xi)))d\xi - \int_a^{\theta(s_1)} K(s_1,\xi,z(\eta(\xi)))d\xi\right|\\
&\le \left|\int_a^{\theta(s_2)} K(s_2,\xi,z(\eta(\xi)))d\xi - \int_a^{\theta(s_2)} K(s_1,\xi,z(\eta(\xi)))d\xi\right|\\
&+ \left|\int_a^{\theta(s_2)} K(s_1,\xi,z(\eta(\xi)))d\xi - \int_a^{\theta(s_1)} K(s_1,\xi,z(\eta(\xi)))d\xi\right|\\
&\le D\omega^1(K,\sigma) + M\omega(\theta,\sigma),
\end{aligned}$$
where $M = \sup\{|K(s,\xi,z)| : s \in I, \xi \in [0,D], z \in [-r_0, r_0]\}\}$ and
$$\omega^1(K,\sigma) = \sup\{|K(s,\xi,z) - K(\bar{s},\xi,z)| : |s - \bar{s}| \le \sigma, s, \bar{s} \in I, \xi \in [0,D], z \in [-r_0, r_0]\}.$$
Thus, we have $\lim_{\sigma\to 0} \omega(\Phi(z),\sigma) = 0$, so $\Phi \in \mathfrak{B}_{r_0}^{\mathrm{C}}(E)$ (see also Example 3 of [32, Section 2]).

(2) With the same assumptions in part (1), let $\Phi \in \mathfrak{B}_{r_0}(E)$ be defined as follows

$$\Phi(z)(s) = \int_0^{\theta(s)} \frac{K(s,\xi,z(\eta(\xi)))}{(\theta(s)-\xi)^{1-\tau}} d\xi.$$

Without loss of generality assume that $|s_2 - s_1| \leq \sigma$ and $\theta(s_1) \geq \theta(s_2)$, then we have

$$\begin{aligned}
|\Phi_1(z)(s_2) - \Phi_1(z)(s_1)| &= \left| \int_0^{\theta(s_2)} \frac{K(s_2,\xi,z(\eta(\xi)))}{(\theta(s_2)-\xi)^{1-\tau}} d\xi - \int_0^{\theta(s_1)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_1)-\xi)^{1-\tau}} d\xi \right| \\
&\leq \left| \int_0^{\theta(s_2)} \frac{K(s_2,\xi,z(\eta(\xi)))}{(\theta(s_2)-\xi)^{1-\tau}} d\xi - \int_0^{\theta(s_2)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_2)-\xi)^{1-\tau}} d\xi \right| \\
&+ \left| \int_0^{\theta(s_2)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_2)-\xi)^{1-\tau}} d\xi - \int_0^{\theta(s_2)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_1)-\xi)^{1-\tau}} d\xi \right| \\
&+ \left| \int_0^{\theta(s_2)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_1)-\xi)^{1-\tau}} d\xi - \int_0^{\theta(s_1)} \frac{K(s_1,\xi,z(\eta(\xi)))}{(\theta(s_1)-\xi)^{1-\tau}} d\xi \right|.
\end{aligned}$$

After some calculations we get

$$|\Phi_1(z)(s_2) - \Phi_1(z)(s_1)| \leq \frac{D}{\tau}\omega^1(K,\sigma) + \frac{M}{\tau}\left[\theta(s_1)^\tau - \theta(s_2)^\tau + (\theta(s_1)-\theta(s_2))^\tau\right].$$

The above inequality shows that $\Phi \in \mathfrak{B}^{\mathrm{C}}_{r_0}(E)$.

(3) With the same assumptions in part (1), let $\Phi \in \mathfrak{B}_{r_0}(E)$ be defined as follows

$$\Phi(z)(s) = \int_0^{\theta(s)} K(s,\xi,z(\eta(\xi))) dB(\xi)$$

where the integral "$\int$" stand for stochastic integral and B is a Brownian motion, see [24] for definition and further results, also, in this paper we assume that Brownian motion is standard, i.e., $B(0) = 0$. Very similar to case (1) one can prove that $\Phi \in \mathfrak{B}^{\mathrm{C}}_{r_0}(E)$.

The following case shows that one can create new functional equations in practice and check the existence of solutions.

Corollary 2.4. With the same assumptions in Examples 2.3-(1)-(3), assume that there exist $r_0 > 0$, non-negative constants $k_j, j = 1,2,3$, and $\phi \in \Omega$ such that $\zeta \in C(I\times[-r_0,r_0]\times[-DK,DK]\times[-\frac{DK}{\Gamma(\tau+1)},\frac{DK}{\Gamma(\tau+1)}]\times[-KD,DK],\mathbb{R})$ satisfies

$$|\zeta(s,u,v_1,v_2,v_3) - \zeta(s,\bar{u},\bar{v}_1,\bar{v}_2,\bar{v}_3)| \leq \phi(|u-\bar{u}|) + \sum_{i=1}^{3} k_i|v_i - \bar{v}_i|, \tag{9}$$

for all $s \in I, u, \bar{u}, \in [-r_0, r_0], v_1, \bar{v}_1, v_3, \bar{v}_3 \in [-DK, DK], , v_2, \bar{v}_2 \in [-\frac{DK}{\Gamma(\tau+1)}, \frac{DK}{\Gamma(\tau+1)}]$, where $K_1, K_2, K_3 \in C(I \times [0,D] \times [0,r_0], \mathbb{R})$ and

$$K = \sup\{|K_1(x,y,u)|, |K_2(x,y,u)|, |K_3(t,s,u)| : u \in [-r_0,r_0], x,y \in I\}.$$

Moreover, let

$$\phi(r_0) + DK\left(2 + \frac{1}{\Gamma(\tau+1)}\right) + M_\zeta \le r_0, \tag{10}$$

where $M_\zeta = \sup\{|\zeta(s,0,0,0,0)| : s \in I\}$ and

$$Tz(s) = \zeta\left(s, z(s), \int_0^{\theta(s)} K_1(s,\xi,z(\eta(\xi)))d\xi, \frac{1}{\Gamma(\tau)}\int_0^{\theta(s)} \frac{K_2(s,\xi,z(\eta(\xi)))}{(\theta(s)-\xi)^{1-\tau}}d\xi, \int_0^{\theta(s)} K_3(s,\xi,z(\eta(\xi)))dB(\xi)\right), \tag{11}$$

for all $s \in I, z \in B_{r_0}(E)$. Then T has a fixed point in $B_{r_0}(E)$.

Proof. It is clear that the functionals $\Psi_1, \Phi_1, \Phi_2, \Phi_3$ are continuous from $C(I)$ into itself,

$$\begin{cases} \Psi_1(z)(s) = z(s), s \in I, \\ \Phi_1(z)(s) = \int_0^{\theta(s)} K_1(s,\xi,z(\eta(\xi)))d\xi, s \in I, \\ \Phi_2(z)(s) = \dfrac{1}{\Gamma(\tau)}\displaystyle\int_0^{\theta(s)} \frac{K_2(s,\xi,z(\eta(\xi)))}{(\theta(s)-\xi)^{1-\tau}}d\xi, s \in I, \\ \Phi_3(z)(s) = \int_0^{\theta(s)} K_3(s,\xi,z(\eta(\xi)))dB(\xi), s \in I. \end{cases}$$

We have $\Psi_1 \in \mathfrak{L}_\rho(E)$ and Examples 2.3-(1)-(3) show that $\Phi_j \in \mathfrak{B}_\rho^{\mathrm{C}}(E), j = 1,2,3$. Thus, the functional

$$Tz(s) = \zeta\left(s, \Psi_1(z)(s), \Phi_1(z)(s), \Phi_2(z)(s), \Phi_3(z)(s)\right), \quad s \in I, z \in B_{r_0}(E),$$

is of the form (1) for $n = 1, m = 3$. It is easy to check that $M_1 = r_0, N_1 = N_3 = KD, N_2 = \frac{DK}{\Gamma(\tau+1)}$ and condition (II) holds. Let $S := I \times [-r_0, r_0] \times [-DK, DK] \times \left[-\frac{DK}{\Gamma(\tau+1)}, \frac{DK}{\Gamma(\tau+1)}\right] \times [-DK, DK]$. Then from (10) and similar to Remark 2.2-(3) we have

$$\begin{aligned} &\sup\{|\zeta(s,u,v_1,v_2,v_3)| : (s,u,v_1,v_2,v_3) \in S\} \\ &\quad \le \sup\{|\zeta(s,u,v_1,v_2,v_3) - \zeta(s,0,0,0,0)| : (s,u,v_1,v_2,v_3) \in S\} \\ &\quad + \sup\{|\zeta(s,0,0,0,0)| : s \in I\} \\ &\quad \le \sup\{\phi(u) + k_1|v_1| + k_2|v_2| + k_3|v_3| : (s,u,v_1,v_2,v_3) \in S\} + M_\zeta \\ &\quad \le \phi(r_0) + DK\left(2 + \frac{1}{\Gamma(\tau+1)}\right) + M_\zeta \le r_0. \end{aligned}$$

Thus, condition (III) holds too. □

2.1 Case study

In this section, we see that some main theorems in the literature can be obtained or improved from Theorem 2.1 as a corollary.

Example 2.5. Let us consider functional equation considered in [27] (see also [7, Subsection 2.6.2])

$$z(s) = q\left(s, z(s), \psi\left(\int_a^T \frac{g'(t)}{(g(s)-g(t))^{1-\tau}} h(s,t,z(t))dt\right)\right) \tag{12}$$

where $\tau \in (0,1), T > a \ge 0$, $\psi : \mathbb{R} \to \mathbb{R}, q : [a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g : [a,T] \to \mathbb{R}$ and $h : [a,T] \times [a,T] \times \mathbb{R} \to \mathbb{R}$ are continuous. The function $g : [a,T] \to \mathbb{R}$ is nondecreasing and with the continuous first derivative. The existence of a solution to Eq. (12) was studied in [27] under the following assumptions:

(A1) $\exists \ell_\psi, C_\psi \geq 0; |\psi(t)-\psi(s)| \leq C_\psi |t-s|^{\ell_\psi}, t, s \in \mathbb{R}$;

(A2) $\exists C_q \geq 0, |q(s,u,v)-q(s,u',v')| \leq \phi(|u-u'|)+C_q|v-v'|, (s,u,v),(s,u',v') \in [a,T]\times\mathbb{R}\times\mathbb{R}$, where $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function such that $\forall t>0$, $\lim_{n\to\infty}\phi_n(t)=0$, where $\phi_n(t)=\phi_{n-1}(\phi(t))$ (Note that this condition yields $\phi(t)<t$ (see [1]));

(A3) There exists $r_0>0$ such that

$$\phi(r_0)+C_qC_\phi\left(\frac{H}{\tau}\right)^{\ell_\phi}(g(T)-g(a))^{\tau\ell_\phi}+M_q+C_q|\phi(0)| \leq r_0,$$

where $H := \sup\{|h(t,s,z(s))| : t,s \in [a,T], z \in C([a,T];\mathbb{R})\} < +\infty$, and $M_q := \max\{|q(t,0,0)| : t \in [a,T]\}$.

Proof. It is clear that the functionals Ψ, Φ are continuous from $C(I)$ into itself.

$$\begin{cases} \Psi(z)(s)=z(s), s\in I \\ \Phi(z)(s)=\psi\left(\int_a^T \frac{g'(t)}{(g(s)-g(t))^{1-\tau}}h(s,t,z(t))dt\right),\ s\in I. \end{cases}$$

Thus, the functional

$$Tz(s)=q\left(s,\Psi(z)(s),\Phi(z)(s)\right),\ s\in I, z\in B_{r_0}(E),$$

is of the form (1). We have $\Psi\in\mathfrak{L}_\rho(E)$, $\Phi\in\mathfrak{B}^{\mathrm{C}}_{r_0}(E)$ and

$$\begin{aligned}
&\sup_{s\in I, z\in B_{r_0}}|\Psi(z)(s)| = \sup_{s\in I, z\in B_{r_0}}|z(s)| \leq r_0,\\
&\sup_{s\in I, z\in B_{r_0}}|\Phi(z)(s)| = \sup_{s\in I, z\in B_{r_0}}\left|\psi\left(\int_a^T \frac{g'(t)}{(g(s)-g(t))^{1-\tau}}h(s,t,z(t))dt\right)\right|\\
&\leq \sup_{s\in I, z\in B_{r_0}}\left|\psi\left(\int_a^T \frac{g'(t)}{(g(s)-g(t))^{1-\tau}}h(s,t,z(t))dt\right)-\psi(0)\right|+|\psi(0)|\\
&\leq C_\psi\left|\int_a^T \frac{g'(t)}{(g(s)-g(t))^{1-\tau}}h(s,t,z(t))dt\right|^{\ell_\psi}+|\psi(0)|\\
&\leq C_\psi\left(\frac{H}{\tau}\right)^{\ell_\phi}(g(T)-g(a))^{\tau\ell_\phi}+|\psi(0)|.
\end{aligned}$$

Then for $M=r_0, N=C_\psi\left(\frac{H}{\tau}\right)^{\ell_\phi}(g(T)-g(a))^{\tau\ell_\phi}+|\psi(0)|$, from (A2)-(A3) and similar to Remark 2.2-(3) we have

$$\begin{aligned}
&\sup\{|q(s,u,v)| : s\in I, u\in[-M,M], v_1\in[-N,N]\}\\
&\quad\leq \sup\{|q(s,u,v)-q(s,0,0)| : s\in I, u\in[-M,M], v\in[-N,N]\}\\
&\quad+\sup\{|q(s,0,0)| : s\in I\}\\
&\quad\leq \sup\{\phi(u)+C_qv : s\in I, u\in[-M,M], v\in[-N,N]\}+M_q\\
&\quad\leq \phi(r_0)+C_q\left[C_\psi\left(\frac{H}{\tau}\right)^{\ell_\phi}(g(T)-g(a))^{\tau\ell_\phi}+|\psi(0)|\right]+M_q\\
&\quad=\phi(r_0)+C_qC_\phi\left(\frac{H}{\tau}\right)^{\ell_\phi}(g(T)-g(a))^{\tau\ell_\phi}+M_q+C_q|\phi(0)| \leq r_0.
\end{aligned}$$

Thus, (I)-(III) hold and equation (12) has a solution in $B_{r_0}(E)$. □

Note that in [27], this conclusion was obtained from another fixed point theorem, and integral equation (12) includes Hadamard-type fractional integral equation [7, Subsection 2.6.2].

Example 2.6. Kazemi et al. [18] used the following conditions to check the fixed point existence solution of fractional integral equation $z = Tz$, where

$$Tz(s) = \zeta\left(s, \Psi_1(z)(s), \Psi_2(z)(s), \Phi_1(z)(s)\right), \quad s \in I, z \in B_\rho(E), s \in I := [0, b], \tag{13}$$

$0 < \tau \leq 1$ and

$$\begin{cases} \Psi_1(z)(s) = f(s, z(\alpha(s))), s \in I, \\ \Psi_2(z)(s) = u(s, z(\beta(s))), s \in I, \\ \Phi_1(z)(s) = \dfrac{1}{\Gamma(\tau)} \displaystyle\int_0^{\theta(s)} \frac{p(s, \xi, z(\gamma(\xi)))}{(\theta(s) - \xi)^{1-\tau}} d\xi, s \in I. \end{cases}$$

(K1) $f, u \in C(I \times \mathbb{R}, \mathbb{R})$, $p \in C(I \times [0, D] \times \mathbb{R}, \mathbb{R})$, $\zeta \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\theta : I \to \mathbb{R}^+$, $\gamma : [0, D] \to I$, $\alpha, \beta : I \to I$, are continuous functions such that $\theta(s) \leq D, \quad \forall\ s \in I$.

(K2) $\exists k_i \geq 0,\ i = 1, \ldots, 5$, with $k_1k_4 + k_2k_5 < 1$ such that

$$\begin{aligned} &|\zeta(s, u_1, u_2, u_3) - \zeta(s, \bar{u_1}, \bar{u_2}, \bar{u_3})| \leq k_1|u_1 - \bar{u_1}| + k_2|u_2 - \bar{u_2}| + k_3|u_3 - \bar{u_3}|; \\ &|f(s, z) - f(s, \bar{z})| \leq k_4|z - \bar{z}|; \\ &|u(s, z) - u(s, \bar{z})| \leq k_5|z - \bar{z}|. \end{aligned}$$

(K3) $\exists\ \rho > 0$ such that

$$\sup\{|\zeta(s, u_1, u_2, u_3)| : s \in I, u_1, u_2 \in [-\rho, \rho], u_3 \in [-\frac{MD^\tau}{\Gamma(\tau+1)}, \frac{MD^\tau}{\Gamma(\tau+1)}]\} \leq \rho, \tag{14}$$

where $M = \sup\{|p(s, \xi, z)| : \forall z \in [-\rho, \rho], \xi \in [0, D], s \in I\}$.

Proof. It is clear that the functionals Ψ_1, Ψ_2, Φ_1 are continuous from $C(I)$ into itself and T is of the form (1) for $n = 2, m = 1$. From (K1)-(K2), ζ satisfies in condition (II), where $\phi_1(t) = k_1t, \phi_2(t) = k_2t$ and we have $\Psi_1, \Psi_2 \in \mathfrak{L}_\rho(E)$. From Example 2.3-(2) we have $\Phi_1 \in \mathfrak{B}^{\mathrm{C}}_\rho(E)$. Thus, (II)-(III) hold and equation (13) has a fixed point solution in $B_\rho(E)$. Also, from (II) it is needed to add conditions $u_1 = \sup_{s\in I, t\in[-\rho.\rho]} |f(s, t)| \leq \rho$ and $u_2 = \sup_{s\in I, t\in[-\rho.\rho]} |u(s, t)| \leq \rho$ in (K3), since we have $\sup_{s\in I, z\in B_\rho} |\Psi_1(z)(s)| = \sup_{s\in I, z\in B_\rho} |f(s, z(\alpha(s)))| \leq \rho$ and $\sup_{s\in I, z\in B_\rho} |\Psi_2(z)(s)| = \sup_{s\in I, z\in B_\rho} |u(s, z(\beta(s)))| \leq \rho$ (see [18] and compare with (4)). □

Example 2.7. Kazemi et al. [20] used the following conditions to check the existence solution of two-dimensional integral equation

$$z(s, t) = q\left(s, t, z(s, t), \int_0^s h(s, t, \zeta, z(\zeta, t))d\zeta, \int_0^s \int_0^t k(s, t, x, y, z(x, y))dydx\right), \tag{15}$$

where $z \in C(I), (s, t) \in I = [0, a] \times [0, b]$ and

(1) $z \in C(I, \mathbb{R}), q \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), k \in C(I \times \mathbb{R}, \mathbb{R}), h \in C(I \times I \times \mathbb{R}, \mathbb{R})$;

(2) There exists a nonnegative constant $0 < c < 1$ such that
$|q(s,t,u,v,w) - q(s,t,\bar{u},\bar{v},\bar{w})| \leq c(|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|)$;

(3) There exists $r_0 \geq 0$ such that q satisfies the following bounded condition

$$\sup\{|q(s,t,u,v,w)| : (s,t) \in I, -r_0 \leq u \leq r_0, -aM_1 \leq v \leq aM_1, -abM_2 \leq w \leq abM_2\} \leq r_0, \tag{16}$$

where
$M_1 = \sup\{|h(s,t,\zeta,u)|; \forall (s,t) \in I \quad \text{and} \quad \zeta \in [0,b], u \in [-r_0, r_0]\}$,
$M_2 = \sup\{|k(s,t,x,y,u)|; \forall (s,t),(x,y) \in I \quad \text{and} \quad u \in [-r_0, r_0]\}$.

Proof. It is clear that the functionals Ψ_1, Φ_1, Φ_2 are continuous from $C(I)$ into itself,

$$\begin{cases} \Psi_1(z)(s,t) = z(s,t), (s,t) \in I, \\ \Phi_1(z)(s,t) = \int_0^s h(s,t,\zeta,z(\zeta,t))d\zeta, (s,t) \in I, \\ \Phi_2(z)(s,t) = \int_0^s \int_0^t k(s,t,x,y,z(x,y))dydx, (s,t) \in I. \end{cases}$$

Thus, the functional

$$Tz(s,t) = q\left(s,t,\Psi_1(z)(s,t), \Phi_1(z)(s,t), \Phi_2(z)(s,t)\right), \ (s,t) \in I, z \in B_\rho(E),$$

is of the form (1). From (2), we have $\Psi_1 \in \mathfrak{L}_\rho(E)$. Also similar to Example 2.3-(1) (note that these examples hold for the multidimensional case) it is easy to check that $\Phi_1, \Phi_2 \in \mathfrak{B}^{\mathrm{C}}_{r_0}(E)$. Thus, (I)-(III) hold and equation (15) has a solution in $B_{r_0}(E)$. □

Example 2.8. Deep et al. [11] used the following conditions to check fixed point existence solution of implicit functional of stochastic integral equation $z = Tz$ in product type, where

$$\begin{aligned} &T(z)(s) = T_1(z)(s)T_2(z)(s), \ s \in I := [0,a], z \in C(I), \qquad (17) \\ &T_1(z)(s) = F\left(s, z(\theta_1(s)), \int_0^s p_1(s,t,z(\theta_2(t)))dB(t), \int_0^a p_2(s,t,z(\theta_3(t))dB(t)\right) \\ &T_2(z)(s) = G\left(s, z(\mu_1(s)), \int_0^s q_1(s,t,z(\mu_2(t)))dB(t), \int_0^a q_2(s,t,z(\mu_3(t)))dB(t)\right), \end{aligned}$$

and the above integrals "$\int$" stand for stochastic integral and B is a Brownian motion. Assume that

(C1) $\theta_1, \theta_2, \theta_3, \mu_1, \mu_2, \mu_3 : I \to I$ are continuous and $F, G \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\exists g > 0$ so that $|F(t,0,0,0)| \leq g$; $|G(t,0,0,0)| \leq g$;

(C2) $h_j : I \to I, j = 1,2,...,6$ are continuous functions and
$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \leq h_1(t)|u_1-u_2| + h_2(t)|v_1-v_2| + h_3(t)|w_1-w_2|$;
$|G(t,u_1,v_1,w_1) - G(t,u_2,v_2,w_2)| \leq h_4(t)|u_1-u_2| + h_5(t)|v_1-v_2| + h_6(t)|w_1-w_2|$;
for all $t \in I$ and $u_1,u_2,v_1,v_2,w_1,w_2 \in \mathbb{R}$;

(C3) $p_1, p_2, q_1, q_2 \in C^1(I \times [0,a] \times \mathbb{R})$;

(C4) $K = \max\{h_j(t) | t \in I\}, j = 1,2,...,6$;

(C5) $\exists \eta, \nu \geq 0$ such that $|p_1(x,y,r)|, |p_2(x,y,r)|, |q_1(x,y,r)|, |q_2(x,y,r)| \leq \eta + \nu|r|$, for all $x,y \in [0,a]$ and $r \in \mathbb{R}$. Further, $4\gamma\delta < 1, \gamma = K + 2K\hat{\zeta}\nu$, where $\delta = g + 2K\hat{\zeta}\eta$ and $\hat{\zeta}$=sup$\{|B(t)| : t \in [0,a]\}$.

Then equation (17) has a solution in E.

Proof. It is clear that the functionals $\Psi_1, \Psi_1', \Phi_1, \Phi_1', \Phi_2, \Phi_2'$ are continuous from $C(I)$ into itself:

$$\begin{cases} u_1 := \Psi_1(z)(s) & = z(\theta_1(s)) \\ v_1 := \Phi_1(z)(s) & = \int_0^s p_1(s,t,z(\theta_2(t)))dB(t), s \in I, v_2 := \Phi_2(z)(s) = \int_0^a p_2(s,t,z(\theta_3(t)))dB(t), s \in I, \\ u_2 := \Psi_1'(z)(s) & = z(\mu_1(s)), s \in I, \\ v_3 := \Phi_1'(z)(s) & = \int_0^s q_1(s,t,z(\mu_2(t)))dB(t), s \in I, v_4 := \Phi_2'(z)(s) = \int_0^a q_2(s,t,z(\mu_3(t)))dB(t), s \in I. \end{cases}$$

Let $\zeta_1 = F, \zeta_2 = G, \rho > 0$. Thus, the functional (17) is of the form

$$Tz(s) = \zeta\left(s, \Psi_1(z)(s), \Psi_1'(z)(s), \Phi_1(z)(s), \Phi_2(z)(s), \Phi_1'(z)(s), \Phi_2'(z)(s)\right), \qquad s \in I, z \in B_\rho(E) \quad (18)$$

where

$$\zeta(s, u_1, u_2, v_1, v_2, , v_3, v_4) = \zeta_1(s, u_1, v_1, v_2)\zeta_2(s, u_2, v_3, v_4).$$

We have $\Psi_1, \Psi_1' \in \mathfrak{L}_\rho(E), \Phi_1, \Phi_2, \Phi_1', \Phi_2' \in \mathfrak{B}_\rho^{\mathrm{C}}(E)$ and from (C5) we have

$$\begin{aligned} N_1 &= \sup\{|\Phi_1(z)(s)| : s \in I, z \in B_\rho(E)\} \\ &= \sup\left\{\left|\int_0^t p_1(t,s,z(\theta_1(s)))dB(s)\right| : s \in I, z \in B_\rho(E)\right\} \le (\eta + \nu||z||)\hat{\zeta}. \end{aligned}$$

Similar calculation shows that $N_2, N_3, N_4 \le (\eta + \nu||z||)\hat{\zeta}$. Let $||z|| \le \rho$ and $N := (\eta + \nu\rho)\hat{\zeta}$. Then we have

$$\begin{aligned} L_1(\rho) = \sup_{z \in B_\rho(E)} ||T_1(z)|| &\le \sup\{|\zeta_1(s, u_1, v_1, v_2)|, s \in I, -\rho \le u_1 \le \rho, N \le v_1, v_2 \le N\} \\ &\le \sup\{|\zeta_1(s, u_1, v_1, v_2)| - \zeta_1(s, 0, 0, 0)| + |\zeta_1(s, 0, 0, 0)|, s \in I, -\rho \le u_1 \le \rho, N \le v_1, v_2 \le N\} \\ &\le K(|u_1| + |v_1| + |v_2|) + f \le K||z|| + 2K((\eta + \nu||z||)\hat{\zeta} + g \\ &= (K + 2K\nu\hat{\zeta})||z|| + 2K\eta\hat{\zeta} + g = \gamma||z|| + \delta \le \gamma\rho + \delta. \end{aligned}$$

By a similar way we have

$$L_2(\rho) = \sup_{z \in B_\rho(E)} ||T_2(z)|| \le \gamma||z|| + \delta \le \gamma\rho + \delta.$$

Thus, we get

$$\begin{aligned} &|\zeta(s, u_1, u_2, v_1, v_2, v_3, v_4) - \zeta(s, \bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{,} \bar{v}_3, \bar{v}_4)| \\ &= |\zeta_1(s, u_1, v_1, v_2)\zeta_2(s, u_2, v_3, v_4) - \zeta_1(s, \bar{u}_1, \bar{v}_1, \bar{v}_2)\zeta_2(s, \bar{u}_2, \bar{v}_3, \bar{v}_4)| \\ &\le |\zeta_1(s, u_1, v_1, v_2)\zeta_2(s, u_2, v_3, v_4) - \zeta_1(s, \bar{u}_1, \bar{v}_1, \bar{v}_2)\zeta_2(s, u_2, v_3, v_4)| \\ &+ |\zeta_1(s, \bar{u}_1, \bar{v}_1, \bar{v}_2)\zeta_2(s, u_2, v_3, v_4) - \zeta_1(s, \bar{u}_1, \bar{v}_1, \bar{v}_2)\zeta_2(s, \bar{u}_2, \bar{v}_3, \bar{v}_4)| \\ &\le L_2(\rho)|\zeta_1(s, u_1, v_1, v_2) - \zeta_1(s, \bar{u}_1, \bar{v}_1, \bar{v}_2)| + L_1(\rho)|\zeta_2(s, u_2, v_3, v_4) - \zeta_2(s, \bar{u}_2, \bar{v}_3, \bar{v}_4)| \\ in &\le L_2(\rho)K(|u_1 - \bar{u}_1| + |v_1 - \bar{v}_1| + |v_2 - \bar{v}_2|) + L_1(\rho)K(|u_2 - \bar{u}_2| + |v_3 - \bar{v}_3| + |v_4 - \bar{v}_4|). \end{aligned} \quad (19)$$

Inequality after [11, Relation (17)], i.e.,

$$K(\gamma\rho + \delta) + K(\gamma\rho + \delta) < 1,$$

where $\rho = \frac{1 - 2\gamma\delta - \sqrt{1 - 4\gamma\delta}}{2\gamma^2}$, shows that we have $KL_2(\rho) + KL_1(\rho) < 1$, thus, the inequality (19) shows that $\zeta \in C(I \times \prod_{i=1}^2[-M_i, M_i] \times \prod_{i=1}^4[-N_i, N_i], \mathbb{R})$ satisfies condition (II). Also it is easy to check that $L_1(\rho)L_2(\rho) = (\gamma\rho + \delta)^2 \le \rho$ (see [11]), thus condition (III) holds too. □

Deep et al. [11] obtained this result from another fixed point theorem.

Example 2.9. Kazemi and Yaghoobnia [23] used conditions (H1)-(H3) to check the fixed point existence solution of

$$T(z)(s) = T_1(z)(s)T_2(z)(s), \ s \in I := [0,a], z \in C(I), \tag{20}$$

where

$$\begin{aligned} T_1(z)(s) &= f(s, z(\alpha(s))) \\ &\quad + F\left(s, z(\tau(s)), \int_0^s p_1(s,t,z(\theta_1(t)))dB(t), \int_0^a p_2(s,t,z(\theta_2(t)))dB(t)\right), \\ T_2(z)(s) &= g(s, z(\beta(s))) \\ &\quad + G\left(s, z(\upsilon(s)), \int_0^s q_1(s,t,z(\mu_1(t)))dB(t), \int_0^a q_2(s,t,z(\mu_2(t)))dB(t)\right). \end{aligned} \tag{21}$$

As previous example the above integrals stand for stochastic integral and B is a Brownian motion, see [23] for more details about (H1)-(H3) and continues functions in (21).

Proof. Kazemi and Yaghoobnia [23] generalized previous example by Petryshyn's fixed point theorem (see [23, Corollary 3.2]). It is clear that the functionals $\Psi_1, \Psi_1', \Psi_2, \Psi_2', \Phi_1, \Phi_1', \Phi_2, \Phi_2'$ are continuous from $C(I)$ into itself:

$$\begin{cases} u_1 := \Psi_1(z)(s) = z(\alpha(s)), u_1' := \Psi_2(z)(s) = z(\tau(s)), s \in I, \\ v_1 := \Phi_1(z)(s) = \int_0^t p_1(s,t,z(\theta_1(t)))dB(t), s \in I, v_2 := \Phi_2(z)(s) = \int_0^t p_1(s,t,z(\theta_1(t)))dB(t), s \in I, \\ u_2 := \Psi_1'(z)(s) = z(\beta(s)), u_2' := \Psi_2'(z)(s) = z(\upsilon(s)), s \in I, \\ v_3 := \Phi_1'(z)(s) = \int_0^t q_1(s,t,z(\mu_1(t)))dB(t), v_4 := \Phi_2'(z)(s) = \int_0^a q_2(s,t,z(\mu_2(t)))dB(s), s \in I. \end{cases}$$

Put $\zeta_1(s,u_1,u_1',v_1,v_2) = f(s,u_1)+F(s,u_1',v_1,v_2), \zeta_2(s,u_2,u_2',v_3,v_4) = g(s,u_2')+G(s,u_2,u_2',v_3,v_4)$. Thus, the functional (20) is of the form (1) where $\zeta = \zeta_1\zeta_2$ and

$$\zeta(s,u_1,u_1',u_2,u_2',v_1,v_2,v_3,v_4) = \zeta_1(s,u_1,u_1',v_1,v_2)\zeta_2(s,u_2,u_2',v_3,v_4).$$

There is a mistake in their proof. They showed that if T_1 and T_2 are densifying maps, then $T = T_1T_2$ is a densifying map too, more precisely, it has been shown that $\chi(T_1(A)) \leq (c+k)\chi(A), \chi(T_2(A)) \leq (c'+k')\chi(A)$, for all bounded sets $A \subset E$, and then it is concluded that T is a densifying map, which is not correct (see [7, Sec 2.5.7], [11, Theorem 2.2] and [4, 5, 8] and previous example). If one adds assumption "there exist $r_0 > 0$ such that $(c+k)(A_2+B_2)+(c'+k')(A_1+B_1) < 1$" to conditions (H1)-(H3) in [23], then similar to previous example it can be proved that $\zeta \in C(I \times \prod_{i=1}^4[-\rho,\rho] \times \prod_{i=1}^2[-A_1 - B_1, A_1+B_1] \times \prod_{i=1}^2[-A_2-B_2, A_2+B_2], \mathbb{R})$ satisfies in condition (II) where $\rho = r_0$ and condition (H3) yields (III), thus, the main result in [23] follows from Theorem 2.1 under some corrections. $\square$

In [22], Kazemi et al. used conditions (H1)-(H3) to check the fixed point existence solution of functional equation as $z = T(z) = T_1(z)T_2(z), z \in C(I)$, where T_1 and T_2 are of the form (13). As in the previous two examples, the result of existence can be concluded.

3 Conclusion

The above examples show that many results in the existence of fixed points of implicit integral functional equations have a similar structure in the proofs. Also, one can combine functional lists (as in Corollary 2.4) to form an integral equation under appropriate conditions that satisfy conditions (II)-(III) and obtain a fixed-point existence result about (integral) functional equations in $C(I)$. Since Theorem 2.1 works for every bounded cube $I \subset \mathbb{R}^r$, one can obtain a multidimensional version of the above examples, for instance, Example 2.7 is a two-dimensional case of [21] with a few changes (see also [14]). Many other results, such as Hadamard-type fractional integral equations, fractional stochastic integral equations (even in product type) and so on, can be obtained in this way, for instance, some of them are [11–15, 17–21, 23, 25, 31]. The interested researchers can think about Eq. (1) on different Banach function spaces, e.g., Orlicz spaces, Lebesgue spaces, bounded variation spaces, Sobolev spaces, etc., by using the concept of superposition operators (see [3]).

References

[1] R. P. Agarwal, E. Karapınar, D. O'Regan, and A. F. Roldán-López-de Hierro. *Fixed point theory in metric type spaces*. Springer, Cham, 2015. ISBN 978-3-319-24080-0; 978-3-319-24082-4. URL `https://doi.org/10.1007/978-3-319-24082-4`.

[2] R. P. Agarwal, M. Meehan, and D. O'Regan. *Fixed point theory and applications*, volume 141 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. ISBN 0-521-80250-4. URL `https://doi.org/10.1017/CBO9780511543005`.

[3] J. Appell and P. P. Zabrejko. *Nonlinear superposition operators*, volume 95 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-36102-8. URL `https://doi.org/10.1017/CBO9780511897450`.

[4] J. Banaś and S. Dudek. The technique of measures of noncompactness in Banach algebras and its applications to integral equations. *Abstr. Appl. Anal.*, 2013: Article ID 537897, 2013. URL `https://doi.org/10.1155/2013/537897`.

[5] J. Banaś and M. Lecko. Fixed points of the product of operators in Banach algebra. *PanAmer. Math. J.*, 12(2):101–109, 2002.

[6] J. Banaś and M. Mursaleen. *Sequence spaces and measures of noncompactness with applications to differential and integral equations*. Springer, New Delhi, 2014. ISBN 978-81-322-1885-2; 978-81-322-1886-9. URL `https://doi.org/10.1007/978-81-322-1886-9`.

[7] J. Banaś, M. Jleli, M. Mursaleen, B. Samet, and C. Vetro, editors. *Advances in nonlinear analysis via the concept of measure of noncompactness*. Springer, Singapore, 2017. ISBN 978-981-10-3721-4; 978-981-10-3722-1. URL `https://doi.org/10.1007/978-981-10-3722-1`.

[8] K. Ben Amara, M. I. Berenguer, and A. Jeribi. Approximation of the fixed point of the product of two operators in banach algebras with applications to some functional equations. *Math.*, 10(22): 4179, 2022.

[9] D. W. Boyd and J. S. W. Wong. On nonlinear contractions. *Proc. Amer. Math. Soc.*, 20:458–464, 1969. URL `https://doi.org/10.2307/2035677`.

[10] Y. J. Cho, M. Jleli, M. Mursaleen, B. Samet, and C. Vetro, editors. *Advances in metric fixed point theory and applications*. Springer, Singapore, 2021. ISBN 978-981-33-6646-6; 978-981-33-6647-3. URL `https://doi.org/10.1007/978-981-33-6647-3`.

[11] A. Deep, S. Abbas, B. Singh, M. Alharthi, and K. S. Nisar. Solvability of functional stochastic integral equations via Darbo's fixed point theorem. *Alex. Eng. J.*, 60(6):5631–5636, 2021.

[12] A. Deep, D. Deepmala, and R. Ezzati. Application of Petryshyn's fixed point theorem to solvability for functional integral equations. *Appl. Math. Comput.*, 395, 2021. URL https://doi.org/10.1016/j.amc.2020.125878.

[13] A. Deep, Deepmala, and B. Hazarika. An existence result for Hadamard type two dimensional fractional functional integral equations via measure of noncompactness. *Chaos Solitons Fractals*, 147: Article ID 110874, 2021. URL https://doi.org/10.1016/j.chaos.2021.110874.

[14] A. Deep, D. Dhiman, B. Hazarika, and S. Abbas. Solvability for two dimensional functional integral equations via Petryshyn's fixed point theorem. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 115(4), Paper No 160, 2021. URL https://doi.org/10.1007/s13398-021-01100-9.

[15] A. Deep, A. Kumar, S. Abbas, and B. Hazarika. An existence result for functional integral equations via Petryshyn's fixed point theorem. *J. Integral Equ. Appl.*, 34(2):165–181, 2022. URL https://doi.org/10.1216/jie.2022.34.165.

[16] M. Furi and A. Vignoli. A fixed point theorem in complete metric spaces. *Boll. Un. Mat. Ital. (4)*, 2:505–509, 1969.

[17] M. Kazemi, H. Chaudhary, and A. Deep. Existence and approximate solutions for Hadamard fractional integral equations in a Banach space. *J. Integral Equ. Appl.*, 35(1):27–40, 2023. URL https://doi.org/10.1216/jie.2023.35.27.

[18] M. Kazemi, A. Deep, and J. Nieto. An existence result with numerical solution of nonlinear fractional integral equations. *Math. Methods Appl. Sci.*, 46(9):10384–10399, 2023. URL https://doi.org/10.1002/mma.9128.

[19] M. Kazemi, A. Deep, and A. Yaghoobnia. Application of fixed point theorem on the study of the existence of solutions in some fractional stochastic functional integral equations. *Math. Sci.*, pages 1–12, 2022. URL https://doi.org/10.1007/s40096-022-00489-7.

[20] M. Kazemi and R. Ezzati. Existence of solution for some nonlinear two-dimensional Volterra integral equations via measures of noncompactness. *Appl. Math. Comput.*, 275:165–171, 2016. URL https://doi.org/10.1016/j.amc.2015.11.066.

[21] M. Kazemi and R. Ezzati. Existence of solutions for some nonlinear volterra integral equations via Petryshyn's fixed point theorem. *Int. J. Nonlinear Anal. Appl.*, 9(1):1–12, 2018.

[22] M. Kazemi, R. Ezzati, and A. Deep. On the solvability of non-linear fractional integral equations of product type. *J. Pseudo-Differ. Oper. Appl.*, 14(3), 2023. URL https://doi.org/10.1007/s11868-023-00532-8.

[23] M. Kazemi and A. R. Yaghoobnia. Application of fixed point theorem to solvability of functional stochastic integral equations. *Appl. Math. Comput.*, 417:Paper No. 126759, 11, 2022. URL https://doi.org/10.1016/j.amc.2021.126759.

[24] F. C. Klebaner. *Introduction to stochastic calculus with applications*. Imperial College Press, London, third edition, 2012. ISBN 978-1-84816-832-9; 1-84816-832-2. URL https://doi.org/10.1142/p821.

[25] F. Mirzaee and N. Samadyar. Extension of darbo fixed-point theorem to illustrate existence of the solutions of some nonlinear functional stochastic integral equations. *Int. J. Nonlinear Anal. Appl.*, 11(1):413–421, 2020.

[26] G. Morosanu. *Functional analysis for the applied sciences*. Universitext. Springer, Cham, 2019. ISBN 978-3-030-27152-7; 978-3-030-27153-4. URL https://doi.org/10.1007/978-3-030-27153-4.

[27] J. J. Nieto and B. Samet. Solvability of an implicit fractional integral equation via a measure of noncompactness argument. *Acta Math. Sci.*, 37(1):195–204, 2017.

[28] W. Petryshyn. Structure of the fixed points sets of k-set-contractions. *Arch. Rational Mech. Anal.*, 40:312–328, 1971.

[29] M. I. Riggio. *Measure of noncompactness, densifying mappings and fixed points*. PhD thesis, Memorial

University of Newfoundland, 1972.

[30] S. Singh, B. Watson, and P. Srivastava. *Fixed point theory and best approximation: the KKM-map principle*, volume 424 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1997. ISBN 0-7923-4758-7. URL https://doi.org/10.1007/978-94-015-8822-5.

[31] H. M. Srivastava, A. Deep, S. Abbas, and B. Hazarika. Solvability for a class of generalized functional-integral equations by means of Petryshyn's fixed point theorem. *J. Nonlinear Convex Anal.*, 22(12): 2715–2737, 2021.

[32] J. M. A. Toledano, T. D. Benavides, and G. L. Acedo. *Measures of noncompactness in metric fixed point theory*, volume 99. Birkhäuser Verlag, Basel, 1997.