A GENERALIZED LOCAL FRACTIONAL LWR MODEL OF VEHICULAR TRAFFIC FLOW AND ITS SOLUTION

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Abstract

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BHAWNA POKHRIYAL AND PRANAY GOSWAMI*,†

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1. Introduction

Controlling traffic on a network of roads to ease congestion and minimize unfavorable side effects (pollution) is one of traffic engineering's goals. It may be necessary to rebuild the network's traffic signs or the roads. An inductive technique termed traffic flow modeling examines patterns in the nature of drivers and vehicles or overall structure of traffic flow. Using the continuum model and continuous functions, traffic flow is effectively depicted. The evolution of traffic states is predicted using continuum models of traffic flow, which are hyperbolic systems that depend only on the initial and boundary conditions. Since they can examine the aggregate behavior of traffic flow using fluid-like state variables like density and flow, these models are also known as macroscopic models. Relationships between the three key variables—flow, density, and velocity, a fundamental characteristic of traffic stream—are necessary for a more realistic depiction of traffic flow [1]. Numerous traffic flow models have been developed over the past few decades as a result of significant research that has been published on the relationship among various traffic flow characteristics.

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Typically, traffic flow models investigate how traffic moves along transport networks. Traffic flow models can be classified as microscopic, mesoscopic, or macroscopic [2]. The visible model simulates traffic flow as a stream of fluid, with the establishment of a flow function and density at every position along a network of roads. In such a model, traffic flow behavior is considered on a global (average) level [3]. At the same time, a microscopic model analyzes the motion of each vehicle. Macroscopic models, called kinematic models, represent traffic flow as a first-order or higher-order continuum like continuous fluid flow [1, 2]. To explain the time-space structures of macroscopic traffic parameters such as vehicle flux, density, and speed, continuum approximations of traffic flow are used [4].
This relates to the remarkable work of Lighthill, Whitham, and Richards, who developed the Lighthill-Whitham-Richards (LWR) model. A macroscopic traffic flow model was separately developed by Lighthill and Whitham(1955) and Richards(1956) to represent the dynamic features of traffic on a homogeneous, unidirectional highway. This model is now referred to as the LWR model in the domain of traffic flow theory. The relationships between three aggregate variables—traffic density, flow rate, and space mean speed—are described in the LWR model, also known as the simple continuum model. The conservation equation is used in the LWR model in the following way:

$$\frac{\partial}{\partial \tau}\Psi(\varepsilon,\tau)+\frac{\partial}{\partial \varepsilon}\varphi(\Psi)=0, \tag{1}$$

where the quantity Ψ represents density in time τ and space ε and quantity φ denotes the vehicle flux as a function of density, Ψ.
Due to its simplicity and potent ability to explain the qualitative behavior of road traffic, the LWR model is still frequently employed to model traffic flow. However, It neglects to investigate well-known aspects of traffic dynamics such as hysteresis, capacity loss, diffusion of platoons, relaxation, or spontaneity in obstruction, for instance, waves like stop and go. A single assumption and two facts form the basis of the LWR model. First, the number of vehicles on a homogeneous route with no sources or sinks is conserved. The Second is the product of density, and speed represents flux. And the underlying presumption is that speed and density has a unique relationship [3, 4]. The PDE form of LWR is used to simulate how queues and shockwaves spread. The LWR model offers a further static relationship between density and flow in the traffic stream. Moreover, because it only requires a small number of model variables, this model is also used for large-scale simulations [5, 6, 7, 8, 9]. A numerical technique was recently used by Tower et al. [10] to study discontinuous velocity version of LWR model. Bürger et al. [11] more recently investigated LWR model with multiclasses that considering a discontinuous velocity function in this series.
Local fractional calculus currently seems a constructive field of applied mathematics in order to investigate the characteristics of fractal space-based physical models. It has been effectively used in several domains, including physics, signal

processing, quantum mechanics, fluid dynamics, applied mathematics, and others, to tackle fractal problems [1, 5]. As a result, this new approach to fractional calculus has been applied by a large number of researchers to the simulation of natural phenomena.. Babakhani and Gejji [12] conducted extensive research on the local fractional derivative, and Yang [13, 14] further developed and refined their findings. Over the past decade, local fractional calculus has become a novel and intriguing theory of fractional calculus that has gained prominence and acceptance among scientists working in this popular branch. Some remarkable work can be seen in references [15, 16, 17, 18]. In the modern era, many problems have been described using local fractional derivatives and integrals, including diffusion and heat equations involving local fractional operators [19], 2-D Burgers-type equations with local fractional derivatives [20], nonlinear local fractional Riccati differential equations [21] and local fractional Burgers equations [22]. References [23, 24, 25] also provide some recent studies on the use of the local fractional natural transform (LFNT) and local fractional Sumudu transform to examined a variety of local fractional equations occurring in the physical sciences. A fractional approach was proposed by Machado and Mata [26] for the bond graph modeling of global economics. Nonlinear singular models were subjected to the stochastic numerical computation technique by Sabir et al [25]. Jafari handled fractional diffusion-wave equations with both linear and nonlinear behavior, and Seifi [27] using the homotopy analysis method. Kumar et al. [28] use the local fractional homotopy perturbation Sumudu transform method (LFHPSTM) to study the local fractional LWR model.

Li et al. [29] used the local fractional Laplace variational iteration approach to analyze the local fractal dynamical LWR model with a highway of finite-length, while Zassim et al. [30] used the local fractional series expansion scheme and the local fractional Laplace decomposition approach.

The standard classical conservation law is violated, rendering the classic LWR model inapplicable when physical parameters such as speed or density in LWR model of vehicle traffic flow are considered as a non-differentiable function in space and time. Therefore, to address this issue, in light of local fractional calculus [15, 16, 17, 18], Wang et al. [31] presented a fractal version of the dynamical LWR model of vehicular traffic flow with local fractional derivatives (LFDs) under the local fractional conservation laws, as stated below:

$$\frac{\partial^\rho}{\partial\tau^\rho}\Psi\left(\varepsilon,\tau\right)+\frac{\partial^\rho}{\partial\varepsilon^\rho}\varphi\left(\Psi\right)=h\left(\varepsilon,\tau\right), \tag{2}$$

subject to initial condition

$$\Psi\left(\varepsilon,0\right)=\Psi_o\left(\varepsilon\right), \quad -\infty<\varepsilon<\infty,\ \tau>0, \tag{3}$$

where the quantity Ψ represents density in time τ and space ε, φ denotes the vehicle flux as function of density and $h\left(\varepsilon,\tau\right)$ denotes source term. The function

$\Psi(\varepsilon,\tau)$ in this instance is a LFC non-differentiable function.
The Cauchy problem of fractal vehicular flow models using local fractional derivatives in linear and nonlinear LWR models was initially examined by Wang et al. [31]. In addition, Guo et al. [32] applied LFVIM to the fractal LWR model and created the entropy criterion for it. The fractal theory provides more accurate estimations of performance measurements than classical derivatives. Consequently, traffic flow for short-term durations can be better predicted using a fractal theory. Following that, a variety of local fractional techniques, including LFVIM [29], local fractional series expansion approach (LFSEA) [30], local fractional Laplace decomposition method (LFLDM) [30], and a hybrid computational scheme [33] has employed to investigate LFLWR model. The local fractional vehicular traffic flow problem was solved by Kumar et al. [34] by combining the Sumudu transform with the local fractional homotopy perturbation approach. All these schemes have been implemented to linear fractional LWR model of vehicular traffic flow.
In this work, a local fractional LWR (LFLWR) model has been extended by considering a special generalized form of the nonlinear fractal LWR model, and the presented nonlinear fractal LWR model with local fractional derivatives has been solved and analyzed by local fractional variational iteration scheme.
The structure of the presented work is as follows: Section 2 gives some brief definitions of the local fractional calculus that are utilised in this work. A generalized nonlinear LFLWR model is presented in Section 3. In Section 4, existence and uniqueness of solution for proposed model is discussed. Section 5 includes an approach of implementing local fractional vatiational iterative scheme and some exemplary instances are discussed in Section 6. Conclusive remarks are covered in Section 7 at the end.

2. Local Fractional Calculus and Properties

In this section, we emphasize the key ideas of the local fractional calculus that is utilized in the presented work.

Definition 2.1. [28, 35, 36] *A function $\Omega(\varepsilon)$, $\varepsilon\in(\mu,\nu)$ is said to be local fractional continuous (LFC) at $\varepsilon=\varepsilon_o$ in (μ,ν) if,*

$$|\Omega(\varepsilon)-\Omega(\varepsilon_o)|<\sigma^{\rho},\ 0<\rho\leq 1, \tag{4}$$

provided, for $\sigma,\delta>0$, $|\varepsilon-\varepsilon_o|<\delta$. And if this is so for all $\varepsilon\in(\mu,\nu)$, then it is LFC on (μ,ν) and denoted as $\Omega(\varepsilon)\in C_{\rho}(\mu,\nu)$.

Definition 2.2. [35, 36] *Let $\Omega(\varepsilon)\in C_{\rho}(\mu,\nu)$, local fractional derivative of $\Omega(\varepsilon)$ at $\varepsilon=\varepsilon_o$ is defined by*

$$D_{\varepsilon}^{\rho}\Omega(\varepsilon_o)=\Omega^{(\rho)}(\varepsilon_o)=\frac{d^{\rho}\Omega(\varepsilon_o)}{d\varepsilon^{\rho}}=\lim_{\varepsilon\to\varepsilon_o}\frac{\Delta^{\rho}(\Omega(\varepsilon)-\Omega(\varepsilon_o))}{\Delta(\varepsilon-\varepsilon_o)^{\rho}}, \tag{5}$$

where

$$\Delta^{\rho}(\Omega(\varepsilon)-\Omega(\varepsilon_o))\cong\Gamma(1+\rho)\,\Omega(\varepsilon)-\Omega(\varepsilon_o). \tag{6}$$

Definition 2.3. [35, 36] *Let the partition of interval* $[\mu,\nu]$ *be* (z_p, z_{p+1}), $p = 0,1,\cdots,M-1$ *and* $z_M = \nu$ *with* $\Delta z_p = z_{p+1} - z_p$ *and* $\Delta z = \max\{\Delta z_0, \Delta z_1, ...\}$. *Then Local fractional integral of* $\Omega(\varepsilon)$ *in* $[\mu,\nu]$ *is defined by*

$$ {}_{\mu}J_{\nu}^{\rho}\Omega(\varepsilon) = \frac{1}{\Gamma(1+\rho)}\int_{\mu}^{\nu}\Omega(z)(dz)^{\rho} = \frac{1}{\Gamma(1+\rho)}\lim_{\Delta z\to 0}\sum_{p=0}^{M-1}\Omega(z_p)(\Delta z_p)^{\rho}. \tag{7}$$

Definition 2.4. [37] *Let* $g : \Re \to \Re$ *be a continuous function. Then integral of* g *w.r.t* $(dt)^{\rho}$ *is defined as*

$$\int_{0}^{\tau} g(t)(dt)^{\rho} = \rho\int_{0}^{\tau}(\tau - t)^{\rho-1} g(t)\, dt,\ 0<\rho\le 1. \tag{8}$$

Definition 2.5. [35, 36] *Let the function* $\Omega : [\mu,\nu]\times R^{\rho} \to R^{\rho}$ *be LFC. Then* Ω *is referred as Lipschitz continuous if* $\exists\, 0<\lambda<1$ *such that for all* $\varepsilon\in[\mu,\nu]$,

$$|\Omega(\varepsilon,\tau_1) - \Omega(\varepsilon,\tau_2)| \le \lambda^{\rho}|\tau_1-\tau_2|,\ 0<\rho\le 1. \tag{9}$$

Definition 2.6. [35] *Let* $B = C_{\rho}$ *be a generalized Banach space (GBS) and* $||\cdot||_{\rho}$ *is a norm defined on space* B. *If a mapping* $\varsigma : B\to B$ *satisfies* $||\varsigma\upsilon^{\rho} - \upsilon^{\rho}||_{\rho} = 0$, $\upsilon^{\rho}\in B$ *then* υ^{ρ} *is said to be fixed point of* ς. *Moreover, if* ς *satisfies*

$$||\varsigma\upsilon^{\rho} - \varsigma\eta^{\rho}||_{\rho} \le \sigma^{\rho}||\upsilon^{\rho} - \eta^{\rho}||_{\rho},\ 0<\sigma<1, \tag{10}$$

for $\eta^{\rho}\in B$, *then* ς *is called a contraction mapping.*

Theorem 2.1. [35] *Let* $(B, ||.||_{\rho})$ *be a complete GBS. For a mapping* $\varsigma : B\to B$, *if* $\exists\,\gamma\ge 1$ *such that* ς^{γ} *is contraction, then the mapping* ς *has a unique fixed point.*

Theorem 2.2. [35] *Let a function* $\Omega : [\mu,\nu]\times R^{\rho}\to R^{\rho}$ *is LFC, then* ς *is Lipschitz continuous.*

Definition 2.7. *The definition of Gamma function,* $\Gamma(\kappa)$, *is given by*

$$\Gamma(\kappa) = \int_{0}^{\infty}\varepsilon^{\kappa-1}\exp(\varepsilon),\ \Re(\kappa)>0. \tag{11}$$

Definition 2.8. [35, 36] *The Mittage Leffler function, Sine function, Cosine function are defined in fractal space as*

$$E_{\rho}(\varepsilon^{\rho}) = \sum_{p=0}^{\infty}\frac{\varepsilon^{p\rho}}{\Gamma(1+p\rho)},\ 0<\rho\le 1, \tag{12}$$

$$\sin_{\rho}(\varepsilon^{\rho}) = \sum_{p=0}^{\infty}(-1)^{p}\frac{\varepsilon^{(2p+1)\rho}}{\Gamma(1+(2p+1)\rho)},\ 0<\rho\le 1, \tag{13}$$

$$\cos_\rho(\varepsilon^\rho) = \sum_{p=0}^{\infty} (-1)^p \frac{\varepsilon^{2p\rho}}{\Gamma(1+2p\rho)},\ 0<\rho\le 1, \tag{14}$$

Following are some significant results:

$$\frac{d^\rho \varepsilon^{p\rho}}{d\varepsilon^\rho} = \frac{\Gamma(1+p\rho)}{\Gamma(1+(p-1)\rho)}\varepsilon^{(p-1)\rho}, \tag{15}$$

$$\frac{d^\rho E_\rho(\varepsilon^\rho)}{d\varepsilon^\rho} = E_\rho(\varepsilon^\rho), \tag{16}$$

$$\frac{d^\rho E_\rho(M\varepsilon^\rho)}{d\varepsilon^\rho} = M E_\rho(M\varepsilon^\rho), \tag{17}$$

$${}_\mu J_\nu^\rho E_\rho(\varepsilon^\rho) = E_\rho(\nu^\rho) - E_\rho(\mu^\rho), \tag{18}$$

$${}_\mu J_\nu^\rho \Omega(\varepsilon) = \Omega(\vartheta)\frac{(\nu-\mu)^\rho}{\Gamma(\rho+1)},\ \vartheta\in(\mu,\nu), \tag{19}$$

$${}_\mu J_\nu^\rho 1 = \frac{(\nu-\mu)^\rho}{\Gamma(\rho+1)}. \tag{20}$$

3. Nonlinear Local Fractal LWR model with special generalized form

With reference to equations (2) and (3), consider $\varphi(\Psi)$ of the form,

$$\varphi(\Psi) = a_o + a_1\Psi^\rho(\varepsilon,\tau) + a_2\Psi^{2\rho}(\varepsilon,\tau) + \cdots + a_n\Psi^{n\rho}(\varepsilon,\tau), \tag{21}$$

where a_i, $i=1,2,...,n$ are constants. Substitute equation (21) in (2), we get,

$$\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau) + \frac{\partial^\rho}{\partial\varepsilon^\rho}\left(a_o + a_1\Psi^\rho(\varepsilon,\tau) + a_2\Psi^{2\rho}(\varepsilon,\tau) + \cdots + a_n\Psi^{n\rho}(\varepsilon,\tau)\right) = h(\varepsilon,\tau). \tag{22}$$

This implies,

$$\begin{aligned}\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau) &+ a_1\Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) + a_2\frac{\Gamma(1+2\rho)}{\Gamma(1+\rho)}\Psi^\rho(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) + \cdots \\ &+ a_n\frac{\Gamma(1+n\rho)}{\Gamma(1+(n-1)\rho)}\Psi^{(n-1)\rho}(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) = h(\varepsilon,\tau).\end{aligned} \tag{23}$$

Thus, we have

$$\begin{aligned}\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau) &+ \eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) + \eta_2\Psi^\rho(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) + \cdots \\ &+ \eta_n\Psi^{(n-1)\rho}(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau) = h(\varepsilon,\tau),\end{aligned} \tag{24}$$

where

$$\eta_i = a_i \frac{\Gamma(1+i\rho)}{\Gamma(1+(i-1)\rho)}, \quad i = 1,2,...,n \tag{25}$$

Thus, Equation (24) represents the nonlinear fractal LWR model with local fractional derivatives.

4. Existence and Uniqueness of Solution of nonlinear LFLWR model

Consider the presented nonlinear LFLWR model in operator form as

$$L_\rho \Psi(\varepsilon,\tau) + N_\rho \Psi(\varepsilon,\tau) = 0, \tag{26}$$

subject to the initial condition,

$$\Psi(\varepsilon,0) = \Psi_o(\varepsilon), \quad -\infty < \varepsilon < \infty,\ \tau > 0, \tag{27}$$

where $L_\rho = \frac{\partial^\rho}{\partial \tau^\rho}$, a linear operator and
$N_\rho = \left(\eta_1 + \eta_2 \Psi^\rho(\varepsilon,\tau) + ... + \eta_i \Psi^{(i-1)\rho}(\varepsilon,\tau)\right) \frac{\partial^\rho}{\partial \varepsilon^\rho}$, a nonlinear operator.
In view of (26), we have

$$L_\rho \Psi(\varepsilon,\tau) = \Theta(\Psi(\varepsilon,\tau)), \tag{28}$$

where

$$\Theta(\Psi(\varepsilon,\tau)) = -\left(\eta_1 + \eta_2 \Psi^\rho(\varepsilon,\tau) + ... + \eta_i \Psi^{(i-1)\rho}(\varepsilon,\tau)\right) \frac{\partial^\rho}{\partial \varepsilon^\rho} \Psi(\varepsilon,\tau). \tag{29}$$

Theorem 4.1. *Suppose that the function* $\Theta(\Psi(\varepsilon,\tau)) = -\left(\eta_1 + \eta_2 \Psi^\rho(\varepsilon,\tau) + ... + \eta_i \Psi^{(i-1)\rho}(\varepsilon,\tau)\right) \frac{\partial^\rho}{\partial \varepsilon^\rho} \Psi(\varepsilon,\tau)$ *is LFC and satisfies Lipschitz continuity, that is,*

$$|\Theta(\Psi_1(\varepsilon,\tau)) - \Theta(\Psi_2(\varepsilon,\tau))| \le \lambda^\rho |\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)|,\ 0 < \rho \le 1,\ 0 < \lambda < 1, \tag{30}$$

then the system

$$L_\rho \Psi(\varepsilon,\tau) = \Theta(\Psi(\varepsilon,\tau)),$$

subject to initial condition

$$\Psi(\varepsilon,0) = \Psi_o(\varepsilon),$$

has a unique solution in $C_\rho(\mu,\nu)$

Proof. Let us define a mapping $\zeta : C_\rho(\mu,\nu) \to C_\rho(\mu,\nu)$ as

$$\zeta(\Psi(\varepsilon,\tau)) = \Psi_o(\varepsilon) + \frac{1}{\Gamma(1+\rho)} \int_\mu^\nu \Theta(\Psi(\varepsilon,\theta))(d\theta)^\rho. \tag{31}$$

We will claim that for $m = 1,2,...$

$$\|\zeta^m(\Psi_1(\varepsilon,\tau)) - \zeta^m(\Psi_2(\varepsilon,\tau))\|_\rho \le \frac{\lambda^{m\rho}|\nu-\mu|^{m\rho}}{\Gamma^m(1+\rho)} \|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho. \tag{32}$$

For $m = 1$,

(33)

$$\|\zeta(\Psi_1(\varepsilon,\tau)) - \zeta(\Psi_2(\varepsilon,\tau))\|_\rho = \left|\frac{1}{\Gamma(1+\rho)}\int_\mu^\nu (\Theta(\Psi_1(\varepsilon,\theta)) - \Theta(\Psi_2(\varepsilon,\theta)))(d\theta)^\rho\right|$$
$$\leq \left|\frac{1}{\Gamma(1+\rho)}\int_\mu^\nu \lambda^\rho\,|\Psi_1(\varepsilon,\theta) - \Psi_2(\varepsilon,\theta)|(d\theta)^\rho\right|$$
$$\leq \frac{\lambda^\rho}{\Gamma(1+\rho)}\,|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)|\,|\nu-\mu|^\rho.$$

Thus, we have

$$\|\zeta(\Psi_1(\varepsilon,\tau)) - \zeta(\Psi_2(\varepsilon,\tau))\|_\rho \leq \frac{\lambda^\rho|\nu-\mu|^\rho}{\Gamma(1+\rho)}\|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho. \tag{34}$$

Let us assume for $m = k$, that is,

$$\left\|\zeta^k(\Psi_1(\varepsilon,\tau)) - \zeta^k(\Psi_2(\varepsilon,\tau))\right\|_\rho \leq \frac{\lambda^{k\rho}|\nu-\mu|^{k\rho}}{\Gamma^k(1+\rho)}\|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho. \tag{35}$$

Now, for $m = k+1$, we have

(36)

$$\left|\zeta^{k+1}(\Psi_1(\varepsilon,\tau)) - \zeta^{k+1}(\Psi_2(\varepsilon,\tau))\right| \leq \frac{1}{\Gamma(1+\rho)}\int_\mu^\nu \left[\Theta\left(\zeta^k(\Psi_1(\varepsilon,\theta))\right) - \Theta\left(\zeta^k(\Psi_2(\varepsilon,\theta))\right)\right](d\theta)^\rho$$
$$\leq \frac{1}{\Gamma(1+\rho)}\int_\mu^\nu \lambda^\rho\left|\zeta^k(\Psi_1(\varepsilon,\theta)) - \zeta^k(\Psi_2(\varepsilon,\theta))\right|(d\theta)^\rho$$
$$\leq \frac{\lambda^\rho}{\Gamma(1+\rho)}\left|\zeta^k(\Psi_1(\varepsilon,\tau)) - \zeta^k(\Psi_2(\varepsilon,\tau))\right||\nu-\mu|^\rho,\ \tau\in(\mu,\nu)$$
$$\leq \frac{\lambda^{(k+1)\rho}|\nu-\mu|^{(k+1)\rho}}{\Gamma^{(k+1)}(1+\rho)}\|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho.$$

This implies,

(37)

$$\left\|\zeta^{k+1}(\Psi_1(\varepsilon,\tau)) - \zeta^{k+1}(\Psi_2(\varepsilon,\tau))\right\|_\rho \leq \frac{\lambda^{(k+1)\rho}|\nu-\mu|^{(k+1)\rho}}{\Gamma^{(k+1)}(1+\rho)}\|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho.$$

Thus, the assertion is proven, and we have

$$\|\zeta^m(\Psi_1(\varepsilon,\tau)) - \zeta^m(\Psi_2(\varepsilon,\tau))\|_\rho \leq \frac{\lambda^{m\rho}|\nu-\mu|^{m\rho}}{\Gamma^m(1+\rho)}\|\Psi_1(\varepsilon,\tau) - \Psi_2(\varepsilon,\tau)\|_\rho. \tag{38}$$

Therefore, the mapping $\zeta : C_\rho(\mu,\nu) \to C_\rho(\mu,\nu)$ is a contraction. Hence, the system has a unique solution in $C_\rho(\mu,\nu)$. $\square$

5. LFVIM: an Approach

Consider the presented nonlinear LFLWR model

$$\begin{aligned}\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau)+\eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau)+\eta_2\Psi^\rho(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau)+\cdots\\ +\eta_n\Psi^{(n-1)\rho}(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau)=h(\varepsilon,\tau),\end{aligned}\tag{39}$$

subject to initial condition

$$\Psi(\varepsilon,0)=\Psi_o(\varepsilon),\quad -\infty<\varepsilon<\infty,\ \tau>0,\tag{40}$$

where $h(\varepsilon,\tau)$ is the non-differentiable source term. Two cases are taken into consideration in this instance.

Case:1 When $h(\varepsilon,\tau)=0$. Then, According to LFVIM, the local fractional correction functional [14] corresponding to (39) is given as

$$\begin{aligned}\Psi_{n+1}(\varepsilon,\tau)=\Psi_n(\varepsilon,\tau)+\\ {}_0J_\tau^\rho\left\{\zeta^\rho(\varepsilon,\theta)\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+\eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+\eta_2\Psi_n^\rho\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+....+\eta_i\Psi_n^{(i-1)\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}\right)\right\},\end{aligned}\tag{41}$$

where ζ^ρ is defined as the Lagrange fractal multiplier and $\widetilde{\Psi_n}$ is the restricted local fractal variation, that is, $\delta^\rho\widetilde{\Psi_n}=0$.

Now, taking local fractal variation of (41), we get

$$\delta^\rho\Psi_{n+1}=\delta^\rho\Psi_n+{}_0J_\tau^\rho\delta^\rho\left\{\zeta^\rho(\varepsilon,\theta)\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+\eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+\eta_2\Psi_n^\rho\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+....+\eta_i\Psi_n^{(i-1)\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}\right)\right\}.\tag{42}$$

The optimal condition of Ψ_{n+1} is given by [14]

$$\delta^\rho\Psi_{n+1}=0,\tag{43}$$

with reference to (43), we have

$$1+\zeta^\rho(\varepsilon,\theta)\,|_{\theta=\tau}=0,\tag{44}$$

$$(\zeta^\rho)^{(\rho)}=0.\tag{45}$$

Thus, the fractal Lagrange multiplier is given as

$$\zeta^\rho(\varepsilon,\theta)=-1.\tag{46}$$

Therefore, the successive iteration formula is given as

$$\Psi_{n+1}=\Psi_n-{}_0J_\tau^\rho\left\{\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+\eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+\eta_2\Psi_n^\rho\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+....+\eta_i\Psi_n^{(i-1)\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}\right)\right\},\tag{47}$$

along with the initial approximation

$$\Psi_o(\varepsilon,\tau)=\Psi(\varepsilon,0)\,. \tag{48}$$

Hence, the solution is given by

$$\Psi(\varepsilon,\tau)=\lim_{n\to\infty}\Psi_n(\varepsilon,\tau)\,. \tag{49}$$

Case 2: When $h(\varepsilon,\tau)$ is not zero. The successive iteration scheme is given by

(50)

$$\Psi_{n+1}=\Psi_n-{}_0J_\tau^\rho\left\{\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+\eta_1\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+\eta_2\Psi_n^\rho\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}+....+\eta_i\Psi_n^{(i-1)\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\widetilde{\Psi_n}-h(\varepsilon,\theta)\right)\right\},$$

along with the initial approximation

$$\Psi_o(\varepsilon,\tau)=\Psi(\varepsilon,0)\,. \tag{51}$$

Hence, the solution is given by

$$\Psi(\varepsilon,\tau)=\lim_{n\to\infty}\Psi_n(\varepsilon,\tau)\,. \tag{52}$$

6. The Non-Differential Solutions of Non-Linear LFLWR Model

In this section, we illustrate several examples to obtain non-differential solutions for the nonlinear LFLWR Model by LFVIM.

Example 6.1. Consider a nonlinear LFLWR Model

$$\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau)+\frac{\partial^\rho}{\partial\varepsilon^\rho}\left(1+\Psi^\rho(\varepsilon,\tau)\right)=0,\ 0<\rho\le 1, \tag{53}$$

subject to initial condition

$$\Psi(\varepsilon,\tau)=E_\rho(\varepsilon^\rho)\,,\ -\infty<\varepsilon<\infty,\ \tau>0. \tag{54}$$

In view of (47), we have

$$\Psi_{n+1}(\varepsilon,\tau)=\Psi_n(\varepsilon,\tau)-{}_0J_\tau^\rho\left\{\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+\Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_n\right)\right\}, \tag{55}$$

with initial approximation

$$\Psi_o(\varepsilon,\tau)=E_\rho(\varepsilon^\rho)\,. \tag{56}$$

Now, first approximation is given as

(57)

$$\begin{aligned}\Psi_1(\varepsilon,\tau)&=\Psi_o(\varepsilon,\tau)-\frac{1}{\Gamma(1+\rho)}\int_0^\tau\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_o(\varepsilon,\theta)+\Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_o(\varepsilon,\theta)\right)(d\theta)^\rho\\&=E_\rho(\varepsilon^\rho)-\frac{1}{\Gamma(1+\rho)}\int_0^\tau\left(\frac{\partial^\rho}{\partial\theta^\rho}E_\rho(\varepsilon^\rho)+\Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}E_\rho(\varepsilon^\rho)\right)(d\theta)^\rho\\&=E_\rho(\varepsilon^\rho)(1-\tau^\rho)\,.\end{aligned}$$

Second approximation is given as

(58)

$$\begin{aligned}
\Psi_2(\varepsilon,\tau) &= \Psi_1(\varepsilon,\tau) - \frac{1}{\Gamma(1+\rho)} \int_0^{\tau} \left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_1(\varepsilon,\theta) + \Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_1(\varepsilon,\theta) \right)(d\theta)^\rho \\
&= \Psi_1(\varepsilon,\tau) - \frac{1}{\Gamma(1+\rho)} \int_0^{\tau} \left(\frac{\partial^\rho}{\partial\theta^\rho}E_\rho(\varepsilon^\rho)(1-\theta^\rho) + \Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}E_\rho(\varepsilon^\rho)(1-\theta^\rho) \right)(d\theta)^\rho \\
&= E_\rho(\varepsilon^\rho)(1-\tau^\rho) + E_\rho(\varepsilon^\rho)\tau^\rho - \Gamma(1+\rho)E_\rho(\varepsilon^\rho)\left[\frac{\tau^\rho}{\Gamma(1+\rho)} - \frac{\Gamma(1+\rho)\tau^{2\rho}}{\Gamma(1+2\rho)}\right] \\
&= E_\rho(\varepsilon^\rho)\left(1-\tau^\rho+\frac{\Gamma^2(1+\rho)\tau^{2\rho}}{\Gamma(1+2\rho)}\right).
\end{aligned}$$

Third approximation is given as

(59)

$$\begin{aligned}
\Psi_3(\varepsilon,\tau) &= \Psi_2(\varepsilon,\tau) - \frac{1}{\Gamma(1+\rho)} \int_0^{\tau} \left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_2(\varepsilon,\theta) + \Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_2(\varepsilon,\theta) \right)(d\theta)^\rho \\
&= \Psi_2(\varepsilon,\tau) - \frac{1}{\Gamma(1+\rho)} \int_0^{\tau} \left(\frac{\partial^\rho}{\partial\theta^\rho}E_\rho(\varepsilon^\rho)\left(1-\theta^\rho+\frac{\Gamma^2(1+\rho)\theta^{2\rho}}{\Gamma(1+2\rho)}\right) \right)(d\theta)^\rho \\
&\quad - \frac{1}{\Gamma(1+\rho)} \int_0^{\tau} \left(\Gamma(1+\rho)\frac{\partial^\rho}{\partial\varepsilon^\rho}E_\rho(\varepsilon^\rho)\left(1-\theta^\rho+\frac{\Gamma^2(1+\rho)\theta^{2\rho}}{\Gamma(1+2\rho)}\right) \right)(d\theta)^\rho \\
&= E_\rho(\varepsilon^\rho)\left(1+\frac{\Gamma^2(1+\rho)\tau^{2\rho}}{\Gamma(1+2\rho)} - \tau^\rho - \frac{\Gamma^3(1+\rho)\tau^{3\rho}}{\Gamma(1+3\rho)}\right).
\end{aligned}$$

Proceeding in the same manner, we get

$$\Psi_n(\varepsilon,\tau) = E_\rho(\varepsilon^\rho)\left(\sum_{j=0}^{n}\frac{\Gamma^{2j}(1+\rho)\tau^{2j\rho}}{\Gamma(1+2j\rho)} - \sum_{j=0}^{n}\frac{\Gamma^{2j+1}(1+\rho)\tau^{(2j+1)\rho}}{\Gamma(1+(2j+1)\rho)}\right). \tag{60}$$

Therefore, the solution of (53) subject to (54) is given as

$$\begin{aligned}
\Psi(\varepsilon,\tau) &= \lim_{n\to\infty}\Psi_n(\varepsilon,\tau) \\
&= E_\rho(\varepsilon^\rho)\left(\cosh_\rho(\Gamma(1+\rho)\tau^\rho) - \sinh_\rho(\Gamma(1+\rho)\tau^\rho)\right),
\end{aligned} \tag{61}$$

and the corresponding solution graph is represented by Figure 1 with parameter $\rho = ln2/ln3$.

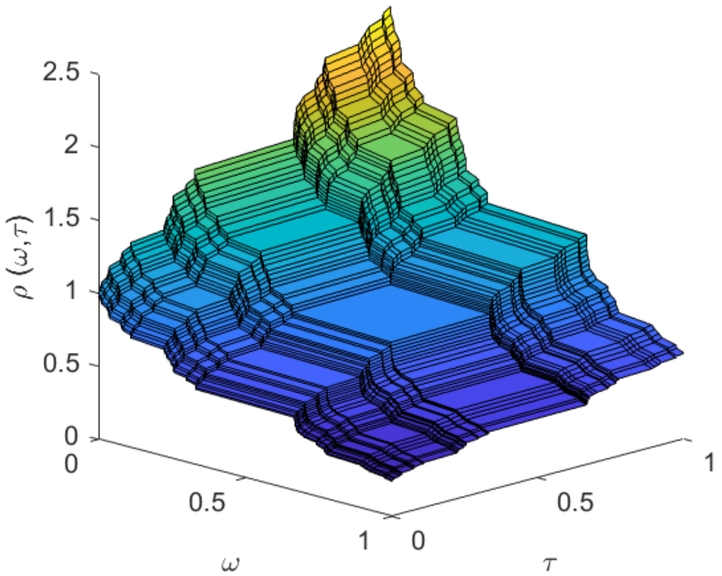


FIGURE 1. Solution graph of (53) subject to(54) with parameter $\rho = ln2/ln3$

Example 6.2. Consider a nonlinear LFLWR Model

$$\frac{\partial^{\rho}}{\partial \tau^{\rho}}\Psi\left(\varepsilon,\tau\right) - \Psi^{\rho}\left(\varepsilon,\tau\right)\frac{\partial^{\rho}}{\partial \varepsilon^{\rho}}\Psi\left(\varepsilon,\tau\right) = 2,\ 0 < \rho \leq 1, \tag{62}$$

subject to initial condition

$$\Psi\left(\varepsilon,0\right) = \frac{\varepsilon^{\rho}}{2},\ -\infty < \varepsilon < \infty,\ \tau > 0. \tag{63}$$

In view of (50), we have

$$\Psi_{n+1} = \Psi_n - {}_0J_{\tau}^{\rho}\left\{\left(\frac{\partial^{\rho}}{\partial \theta^{\rho}}\Psi_n - {\Psi_n}^{\rho}\frac{\partial^{\rho}}{\partial \varepsilon^{\rho}}\Psi_n - 2\right)\right\}, \tag{64}$$

with initial approximation

$$\Psi_o\left(\varepsilon,\tau\right) = \frac{\varepsilon^{\rho}}{2}. \tag{65}$$

Using (64), first approximation is given as follows

$$
\begin{aligned}
\Psi_1 &= \Psi_o - \tfrac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\tfrac{\partial^\rho}{\partial \theta^\rho} \Psi_o - {\Psi_o}^\rho \tfrac{\partial^\rho}{\partial \varepsilon^\rho} \Psi_o - 2 \right) (d\theta)^\rho \\
&= \tfrac{\varepsilon^\rho}{2} - \tfrac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \tfrac{\partial^\rho}{\partial \theta^\rho} \left(\tfrac{\varepsilon^\rho}{2} \right) - \left(\tfrac{\varepsilon^\rho}{2} \right)^\rho \tfrac{\partial^\rho}{\partial \varepsilon^\rho} \left(\tfrac{\varepsilon^\rho}{2} \right) - 2 \right\} (d\theta)^\rho \\
&= \tfrac{\varepsilon^\rho}{2} - \tfrac{1}{\Gamma(1+\rho)} \int_0^\tau \left(-\tfrac{\Gamma(1+\rho)}{2^{\rho+1}} (\varepsilon^\rho)^\rho - 2 \right) (d\theta)^\rho \\
&= \tfrac{\varepsilon^\rho}{2} + \tfrac{(\varepsilon^\rho)^\rho \tau^\rho}{2^{\rho+1}} + \tfrac{2\tau^\rho}{\Gamma^2(1+\rho)}.
\end{aligned} \tag{66}
$$

Similarly, second approximation is given as

(67)

$$
\begin{aligned}
\Psi_2 &= \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\frac{\partial^\rho}{\partial \theta^\rho} \Psi_1 - {\Psi_1}^\rho \frac{\partial^\rho}{\partial \varepsilon^\rho} \Psi_1 - 2 \right) (d\theta)^\rho \\
&= \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \frac{\partial^\rho}{\partial \theta^\rho} \left(\frac{\varepsilon^\rho}{2} + \frac{(\varepsilon^\rho)^\rho}{2^{\rho+1}} + \frac{2\theta^\rho}{\Gamma^2(1+\rho)} \right) \right\} (d\theta)^\rho \\
&- \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \left(\frac{\varepsilon^\rho}{2} + \frac{(\varepsilon^\rho)^\rho}{2^{\rho+1}} + \frac{2\theta^\rho}{\Gamma^2(1+\rho)} \right)^\rho \frac{\partial^\rho}{\partial \varepsilon^\rho} \left(\frac{\varepsilon^\rho}{2} + \frac{(\varepsilon^\rho)^\rho}{2^{\rho+1}} + \frac{2\theta^\rho}{\Gamma^2(1+\rho)} \right) - 2 \right\} (d\theta)^\rho \\
&= \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \frac{2}{\Gamma(1+\rho)} - \frac{\rho\,\Gamma(1+\rho)}{2^{2\rho+1}} (\varepsilon^\rho)^{2\rho-1} \theta^\rho - \frac{\Gamma(1+\rho)}{2^{\rho^2+1}} (\varepsilon^\rho)^{\rho^2} (\theta^\rho)^\rho \right\} (d\theta)^\rho \\
&- \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \frac{\rho\,\Gamma(1+\rho)}{2^{\rho^2+\rho+2}} (\varepsilon^\rho)^{\rho^2+\rho-1} (\theta^\rho)^{\rho+1} - \frac{2^{\rho-1}}{\Gamma^{2\rho-1}(1+\rho)} (\theta^\rho)^\rho \right\} (d\theta)^\rho \\
&- \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \frac{\rho}{2\,\Gamma^{2\rho-1}(1+\rho)} (\varepsilon^\rho)^{\rho-1} (\theta^\rho)^{\rho+1} - 2 \right\} (d\theta)^\rho \\
&= \frac{\varepsilon^\rho}{2} + \frac{(\varepsilon^\rho)^\rho \tau^\rho}{2^{\rho+1}} + \frac{\rho\,\Gamma^2(1+\rho)}{2^{2\rho+1}\Gamma(1+2\rho)} (\varepsilon^\rho)^{2\rho-1} \tau^{2\rho} + \frac{\mathrm{B}(\rho,\rho^2+1)}{2^{\rho^2+2}} (\varepsilon^\rho)^{\rho^2} (\tau^\rho)^{\rho+1} \\
&+ \frac{\rho^2\Gamma(1+\rho)\,\mathrm{B}(\rho,\rho^2+\rho+1)}{2^{\rho^2+\rho+2}} (\varepsilon^\rho)^{\rho^2+\rho-1} (\tau^\rho)^{\rho+2} + \frac{\rho\,2^{\rho-1}\mathrm{B}(\rho,\rho^2+1)}{\Gamma^{2\rho-1}(1+\rho)} (\tau^\rho)^{\rho+1} \\
&+ \frac{\rho^2\mathrm{B}(\rho,\rho^2+\rho+1)}{2\Gamma^{2\rho-1}(1+\rho)} (\varepsilon^\rho)^{\rho-1} (\tau^\rho)^{\rho+2} + \frac{2}{\Gamma(1+\rho)} \tau^\rho,
\end{aligned}
$$

where $\mathrm{B}(\chi,\varsigma) = \frac{\Gamma(\chi)\Gamma(\varsigma)}{\Gamma(\chi+\varsigma)}$ with $\Re(\chi), \Re(\varsigma) > 0$ is the beta function.
Now, proceeding in a similar manner, we get the solution of (62) along with the

condition (63) as

$$\Psi\left(\varepsilon,\tau\right)=\lim_{n\to\infty}\Psi_{n}\left(\varepsilon,\tau\right), \tag{68}$$

and graphs for the above successive approximations are demonstrated by Figure 2 where parameter $\rho = ln2/ln3$. These graphical representations demonstrate the dynamic evolution of the non-differentiable traffic density function $\Psi\left(\varepsilon,\tau\right)$ for the nonlinear LFLWR model of traffic flow that has been stated in the preceding example. It is clear from the presented graphs that these approximated iterations are converging close to the surface of the solution. Bildik and Konuralp [38] consider the classic case, that is, when $\rho = 1$, then the successive approximations converge to its exact solution, which is given as the

$$\Psi\left(\varepsilon,\tau\right)=\frac{\tau^{2}-4\tau-\varepsilon}{\tau-2}. \tag{69}$$

Example 6.3. Consider a nonlinear LFLWR Model

$$\frac{\partial^{\rho}}{\partial\tau^{\rho}}\Psi\left(\varepsilon,\tau\right)-\Psi^{\rho}\left(\varepsilon,\tau\right)\frac{\partial^{\rho}}{\partial\varepsilon^{\rho}}\Psi\left(\varepsilon,\tau\right)=\Gamma\left(1+\rho\right)\left(1-\left(\varepsilon^{\rho}\right)^{\rho}\right),\ 0<\rho\leq 1, \tag{70}$$

subject to initial condition

$$\Psi\left(\varepsilon,0\right)=\varepsilon^{\rho},\ -\infty<\varepsilon<\infty,\ \tau>0. \tag{71}$$

In view of (50), we have

$$\Psi_{n+1}=\Psi_{n}-{}_{0}J_{\tau}^{\rho}\left\{\left(\frac{\partial^{\rho}}{\partial\theta^{\rho}}\Psi_{n}-{\Psi_{n}}^{\rho}\frac{\partial^{\rho}}{\partial\varepsilon^{\rho}}\Psi_{n}-\Gamma\left(1+\rho\right)\left(1-\left(\varepsilon^{\rho}\right)^{\rho}\right)\right)\right\}, \tag{72}$$

with initial approximation

$$\Psi_{o}\left(\varepsilon,\tau\right)=\varepsilon^{\rho}. \tag{73}$$

Now, first approximation is given as

$$\begin{aligned}\Psi_{1}&=\Psi_{0}-\frac{1}{\Gamma\left(1+\rho\right)}\int_{0}^{\tau}\left(\frac{\partial^{\rho}}{\partial\theta^{\rho}}\Psi_{o}-{\Psi_{o}}^{\rho}\frac{\partial^{\rho}}{\partial\varepsilon^{\rho}}\Psi_{o}-\Gamma\left(1+\rho\right)\left(1-\left(\varepsilon^{\rho}\right)^{\rho}\right)\right)(d\theta)^{\rho}\\&=\varepsilon^{\rho}-\frac{1}{\Gamma\left(1+\rho\right)}\int_{0}^{\tau}\left(\frac{\partial^{\rho}}{\partial\theta^{\rho}}\varepsilon^{\rho}-\left(\varepsilon^{\rho}\right)^{\rho}\frac{\partial^{\rho}}{\partial\varepsilon^{\rho}}\varepsilon^{\rho}-\Gamma\left(1+\rho\right)\left(1-\left(\varepsilon^{\rho}\right)^{\rho}\right)\right)(d\theta)^{\rho}\\&=\varepsilon^{\rho}-\frac{1}{\Gamma\left(1+\rho\right)}\int_{0}^{\tau}\left(-\left(\varepsilon^{\rho}\right)^{\rho}\Gamma\left(1+\rho\right)-\Gamma\left(1+\rho\right)\left(1-\left(\varepsilon^{\rho}\right)^{\rho}\right)\right)(d\theta)^{\rho}\\&=\varepsilon^{\rho}+\tau^{\rho}.\end{aligned} \tag{74}$$

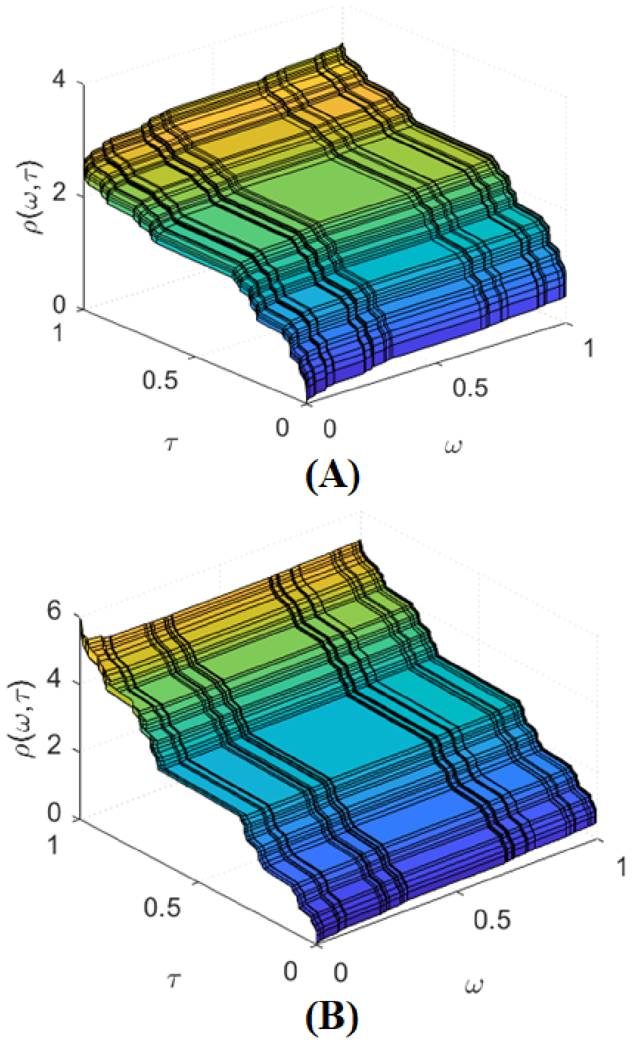


FIGURE 2. (A) and (B) represents first and second approximations of (62) subject to (63) with parameter $\rho = ln2/ln3$

Second approximation is given as

(75)

$$\Psi_2 = \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\frac{\partial^\rho}{\partial \theta^\rho} \Psi_1 - {\Psi_1}^\rho \frac{\partial^\rho}{\partial \varepsilon^\rho} \Psi_1 - \Gamma(1+\rho)\left(1 - (\varepsilon^\rho)^\rho\right) \right) (d\theta)^\rho$$

$$= \Psi_1(\varepsilon, \tau) - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\frac{\partial^\rho}{\partial \theta^\rho} (\varepsilon^\rho + \theta^\rho) - (\varepsilon^\rho + \theta^\rho)^\rho \frac{\partial^\rho}{\partial \varepsilon^\rho} (\varepsilon^\rho + \theta^\rho) \right) (d\theta)^\rho$$

$$+ \frac{1}{\Gamma(1+\rho)} \int_0^\tau \Gamma(1+\rho)\left(1 - (\varepsilon^\rho + \theta^\rho)^\rho\right)(d\theta)^\rho$$

$$= \varepsilon^\rho + \tau^\rho - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\Gamma(1+\rho)\left((\varepsilon^\rho)^\rho - (\varepsilon^\rho + \theta^\rho)^\rho\right)\right)(d\theta)^\rho$$

$$= \varepsilon^\rho + \tau^\rho - \rho\, \mathrm{B}\left(\rho, \rho^2 + 1\right) (\tau^\rho)^{\rho+1},$$

where B represents beta function.
Proceeding in the same way, we get the solution of (70) subject to (71) as

$$\Psi(\varepsilon,\tau)=\lim_{n\to\infty}\Psi_n(\varepsilon,\tau), \tag{76}$$

and graphs for the above iterations are presented by Figure 3 with parameter $\rho = ln2/ln3$.

Example 6.4. Consider a nonlinear LFLWR Model

$$\frac{\partial^\rho}{\partial\tau^\rho}\Psi(\varepsilon,\tau)+\Psi^{2\rho}(\varepsilon,\tau)\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi(\varepsilon,\tau)=\Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho}-1\right),\ 0<\rho\le 1, \tag{77}$$

subject to initial condition

$$\Psi(\varepsilon,0)=\varepsilon^\rho,\ -\infty<\varepsilon<\infty,\ \tau>0. \tag{78}$$

With reference to (50), we have

$$\Psi_{n+1}=\Psi_n-{}_0J_\tau^\rho\left\{\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_n+{\Psi_n}^{2\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_n-\Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho}-1\right)\right)\right\}, \tag{79}$$

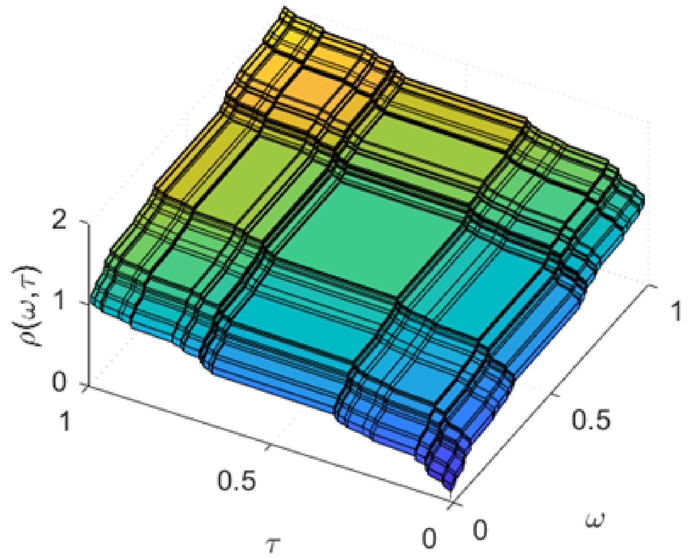
and initial approximation is given as

$$\Psi_o(\varepsilon,\tau)=\varepsilon^\rho. \tag{80}$$

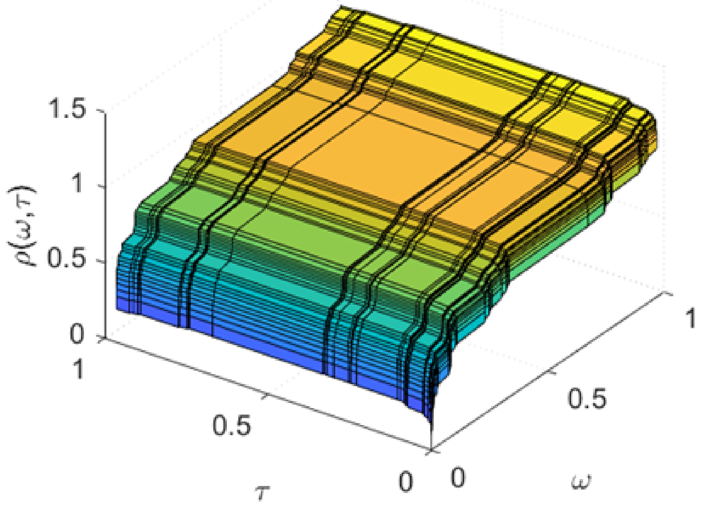
Now, successive approximations are evaluated as

(81)

$$\begin{aligned}\Psi_1&=\Psi_o-\frac{1}{\Gamma(1+\rho)}\int_0^\tau\left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_o+{\Psi_o}^{2\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_o-\Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho}-1\right)\right)(d\theta)^\rho\\&=\varepsilon^\rho-\frac{1}{\Gamma(1+\rho)}\int_0^\tau\left(\frac{\partial^\rho}{\partial\theta^\rho}(\varepsilon^\rho)+(\varepsilon^\rho)^{2\rho}\frac{\partial^\rho}{\partial\varepsilon^\rho}(\varepsilon^\rho)-\Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho}-1\right)\right)(d\theta)^\rho\\&=\varepsilon^\rho-\tau^\rho.\end{aligned}$$



(A)



(B)

FIGURE 3. (A) and (B) represents first and second approximations of (70) subject to (71) with parameter $\rho = ln2/ln3$

(82)

$$\begin{aligned}
\Psi_2 &= \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\frac{\partial^\rho}{\partial\theta^\rho}\Psi_1 + {\Psi_1}^{2\rho} \frac{\partial^\rho}{\partial\varepsilon^\rho}\Psi_1 - \Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho} - 1\right) \right) (d\theta)^\rho \\
&= \Psi_1 - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\frac{\partial^\rho}{\partial\theta^\rho}(\varepsilon^\rho - \theta^\rho) + (\varepsilon^\rho - \theta^\rho)^{2\rho} \frac{\partial^\rho}{\partial\varepsilon^\rho}(\varepsilon^\rho - \theta^\rho) \right) (d\theta)^\rho \\
&+ \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left(\Gamma(1+\rho)\left((\varepsilon^\rho)^{2\rho} - 1\right) \right) (d\theta)^\rho \\
&= \varepsilon^\rho - \tau^\rho - \frac{1}{\Gamma(1+\rho)} \int_0^\tau \left\{ \Gamma(1+\rho)\left[(\varepsilon^\rho - \theta^\rho)^{2\rho} - (\varepsilon^\rho)^{2\rho}\right] \right\} (d\theta)^\rho \\
&= \varepsilon^\rho - \tau^\rho + (\varepsilon^\rho)^{2\rho}\tau^\rho + \frac{1}{2\rho+1}(\varepsilon^\rho - \tau^\rho)^{2\rho+1} + (\varepsilon^\rho)^{2\rho+1}.
\end{aligned}$$

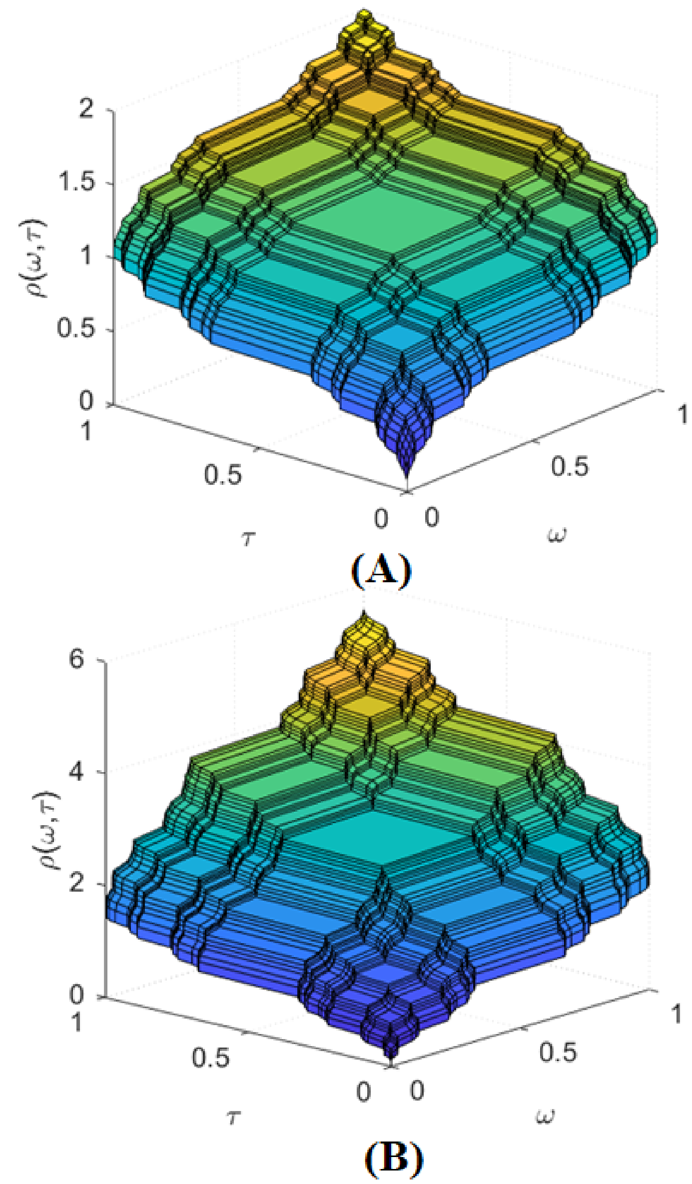


FIGURE 4. (A) and (B) represents first and second approximations of (77) and (78) with parameter $\rho = ln2/ln3$

Continuing in the same way, we get the solution of (77) subject to (78) as

$$\Psi(\varepsilon,\tau) = \lim_{n\to\infty} \Psi_n(\varepsilon,\tau)\,. \tag{83}$$

The graphs for the aforementioned iterations are shown in Figure 4 with parameter $\rho = ln2/ln3$ and demonstrate the dynamic evolution of the non-differentiable traffic density function $\Psi(\varepsilon, \tau)$ for the non-homogeneous nonlinear LFLWR model of traffic flow described in the above example.

7. Conclusion

This research proposed a generalized nonlinear LFLWR model of fractal vehicular traffic flow and non-differentiable solutions are examined by the implementation of LFVIM. Several studies have been conducted in the literature to solve linear LWR and linear LFLWR models, however this research focused on the generalised nonlinear LFLWR model, making it novel. At the first, existence and uniqueness of the solution of suggested model has been discussed. The success of implementing LFVIM to proposed non linear LFLWR model is demonstrated by some illustrative examples. Graphical representations of the successive approximations of solutions have also been shown to demonstrate the dynamic evolution of the non-differentiable traffic density function $\Psi(\varepsilon, \tau)$ for the nonlinear LFLWR model of vehicular traffic flow. Moreover, it is observed from the graphs that these approximations are converging near the solution surface. The computational findings show that the local fractional iterative scheme that has employed is successful and efficient in deriving the non-differentiable solution for generalized nonlinear LFLWR model. The proposed model can be extended to any nonlinear version of LFLWR model of traffic flow as part of the future scope of work and examined through several iterative approaches to uncover fresh insights and findings.

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