A Review of β– Laplace transform of Fractional order of Differential Equations

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Abstract

In this article,study of β-Laplace, β- Laplace-Carson,β- Natural transform of fractional order and some of properties of β-Laplace transform of fractional order mentioned.Further apply β- fractional order Laplace transform on Mittag- Leffler function, Riemann-Liouville integral and Caputo fractional derivatives. Also we obtain a β- inverse Laplace Transform of fractional order.

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RESEARCH ARTICLE

A Review of β- Laplace transform of Fractional order of Differential Equations

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Summary

In this article,study of β-Laplace, β- Laplace-Carson,β- Natural transform of fractional order and some of properties of β- Laplace transform of fractional order mentioned.Further apply β- fractional order Laplace transform on Mittag- Leffler function, Riemann-Liouville integral and Caputo fractional derivatives. Also we obtain a β- inverse Laplace Transform of fractional order.

KEYWORDS:
β- Laplace transform of fractional order,β- Natural Transform, fractional derivatives, Mittag-Leffler Function.

1 | INTRODUCTION

The many more researchers have attempted to create a novel definition for fractional derivative. The vast majority of these definitions contain integral forms for fractional derivatives. In fractional calculus, there are numerous types of differential derivatives, such as Grunwald-Letnikov,Caputo, and Riemann-Liouville, , among others Atangana-Baleanu , Caputo-Fabrizio and more recently one, the conformable fractional derivative (CFD) are examples of previous fractional derivative models.[1,2,3,4,5,6,7,8,9]

The chain rule, a useful and important calculus rule, is applicable only to conformable fractional derivatives. Some authors have recently suggested the notion of non-local derivative. Khalil[10] introduced in a new definition of derivative that is very compatible with the traditional meaning; this operator is known as "conformable derivative." This derivative satisfied a variety of traditional features, such as the chain rule. Conformable differential equations can be resolved with this operator. The Conformable fractional derivative possesses a number of advantageous qualities. As a result, it is now widely used in numerous study domains. Nevertheless, Ortigueira determined that the CFD is not a genuine fractional definition.[11]

It has been found that fractional-order calculus is the best way to describe many physical,chemical,electrical science and engineering science processes (FOC). It is also known that FOC has many advantages over integer-order calculus (IOC). FOC also works where IOC often doesn't work well enough. FOC has many uses in many different fields, such as vibration and control, mechanics, control theory, economics, signal processing ,image processing, fractional Brownian motion, Levy statistics, kinetic model, Riesz potential, power law,electrical engineering, chemical science, life science, geophysics, fractional derivative and fractals, fluid dynamics, bio-medical engineering,computational fractional derivative equations, fractional filters, soft matter mechanics, etc. Natural transform, Laplace transform, Laplace-Carson transform, and other similar methods are very useful in pure and applied mathematics, and they are also closely related.[12,13,14,15,16,17]

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

We have a similar way of introducing the definitions of β-Laplace, β-Laplace-Carson, and β-Natural transform of fractional order. We also try to find some properties of β-Laplace transform of fractional order and show how it relates to standard Laplace transform.[18,19,20]

2 | PRELIMINARY

Basic Notations and Definitions are given.

2.1 | Fractional derivatives[21,22]

Definition 1. If $\xi(c)$ is not necessarily differentiable function but it is contionuous function and then forward operator $FW(p)\xi(c) = \xi(c+p)$ where $p > 0$ indicates a constant . In addition, the fractional difference of $xi(c)$ is called

$$\Delta^{\alpha}\xi(c) = (FW - p)^{\alpha}\xi(c) = \sum_{q=0}^{\infty}(-1)^{q}\begin{pmatrix} \alpha \\ \mathrm{q} \end{pmatrix}\xi\left[c + (\alpha - q)p\right] \tag{1}$$

where, $0 < \alpha < 1$, and $\alpha-$ derivative of $\xi(c)$ is known as

$$\xi(c)^{(\alpha)} = \lim_{p\downarrow 0}\frac{\Delta^{\alpha}\xi(c)}{p^{\alpha}} \tag{2}$$

2.2 | Novel fractional Riemann-Liouville derivative[23,21,22,24]

The novel definition of the R-L fractional derivative recommended by Jumarie(2009).

Definition 2. If $\xi(c)$ is not necessarily differentiable function but it is contionuous function, then $\alpha-$ derivative of $\xi(c)$ is defined as

$$D_c^{\alpha}K = \begin{cases} K\Gamma^{-1}(1-\alpha)c^{-\alpha}, & \alpha \le 0 \\ 0 & otherwise \end{cases} \tag{3}$$

where $\xi(c)$ constant.
On the other hand, when $\xi(c) \neq K$ then $\xi(c) = \xi(0) + (\xi(c) - \xi(0))$.
Fractional derivative of the function $\xi(c)$ will be known as

$$\xi^{(\alpha)}(c) = D_c^{\alpha}\xi(0) + D_c^{\alpha}(\xi(c) - \xi(0))$$

$(\alpha < 0)$ At any negative α, one has

$$D_c^{\alpha}(\xi(c) - \xi(0)) = \frac{1}{\Gamma(-\alpha)}\int_0^c (c-n)^{-\alpha-1}\xi(n)dn, \quad \alpha < 0$$

while for positive α we will put

$$D_c^{\alpha}(\xi(c) - \xi(0)) = D_c^{\alpha}(\xi(c) = D_c(\xi^{(\alpha-1)})$$

when $z < \alpha < z+1$, we place

$$\xi^{(\alpha)}(c) = (\xi^{(\alpha-z)}(c))^{(z)}, z \le \alpha \le z+1, z \ge 1$$

2.3 | Integrad with respected to $(dt)^{\alpha}$

Definition 3. If $\xi(c)$ is not necessarily differentiable function but it is contionuous function, then $\alpha-$ derivative of $\xi(c)$ is defined as

$$D_c^{\alpha}K = \begin{cases} K\Gamma^{-1}(1-\alpha)c^{-\alpha}, & \alpha \le 0 \\ 0 & otherwise \end{cases} \tag{4}$$

where $\xi(c)$ constant.
On the other hand, when $\xi(c) \neq K$ then $\xi(c) = \xi(0) + (\xi(c) - \xi(0))$.
the function with Fractional derivative $\xi(c)$ will be known as

$$\xi^{(\alpha)}(c) = D_c^{\alpha}\xi(0) + D_c^{\alpha}(\xi(c) - \xi(0))$$

At any negative α, ($\alpha < 0$) one has

$$D_c^{\alpha}(\xi(c)-\xi(0)) = \frac{1}{\Gamma(-\alpha)}\int_0^c (c-n)^{-\alpha-1}\xi(n)dn, \quad \alpha < 0$$

while for positive α we will put

$$D_c^{\alpha}(\xi(c)-\xi(0)) = D_c^{\alpha}(\xi(c) = D_c(\xi^{(\alpha-1)})$$

when $z < \alpha < z+1$, we place

$$\xi^{(\alpha)}(c) = (\xi^{(\alpha-z)}(c))^{(z)}, z \leq \alpha \leq z+1, z \geq 1$$

Definition 4. [24] Suppose that the continuous function

$$For, g(z) = z, z \in R, f : R \to R \tag{5}$$

$$g(z+h) = \sum_{k=0}^{\infty} \frac{\lambda h^{\alpha k}}{\alpha k!} f^{\alpha k}(z), 0 < \alpha \leq 1; \tag{6}$$

with the notation $D^{2\alpha}f(z) = D^{\alpha}D^{\alpha}f(z)$.

Definition 5. Integration with respect to $(dz)^{\alpha}$:
[24] The integral with respect to $(dz)^{\alpha}$ is defined as the solution of the fractional differential equation

$$dy = f(z)(dz)^{\alpha}, z \geq 0, y(0) = 0, \tag{7}$$

which is provided by the following result:

Lemma 1. [24] Let f(x) denote a continuous function; then the solution $y(x) = 0$ of equation (7) is defined by the equality

$$\begin{aligned} y &= \int_0^x f(\xi)^{\alpha} \\ &= \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi,\ 0 < \alpha < 1. \end{aligned} \tag{8}$$

3 | MAIN RESULTS

Definition 6. [19,20] If $p(g)$ is a function defined for all $g \geq 0$, then the fractional β- laplace Transform of order α, represented by $\mathcal{L}_{\beta}^{\alpha}\{p(g)\}$ and defined by

$$\begin{aligned} \mathcal{L}_{\beta}^{\alpha}\{p(g)\} = \mathfrak{F}_{\beta}^{\alpha}(s,g) &= \int_0^{\infty} \beta^{-(sg)^{\alpha}} p(g)(dg)^{\alpha}, 0 < \alpha \leq 1 \\ &= \int_0^{\infty} E_{\alpha}(-g^{\alpha}s^{\alpha}(\ln\beta)^{\alpha})p(g)(dg)^{\alpha} \end{aligned} \tag{9}$$

where $s \in \mathbb{C}$ and $E_{\alpha}(x)$ is the mittag- Leffler function $\sum_{n=0}^{\infty} \frac{x^n}{\alpha n!}$, $\beta \in (0,\infty)\backslash\{1\}$

Remark 1. From the above definition (6) we show that

1. When $\beta = e$ we have fractional Laplace Transforms which is proposed in Jumarie(2009).
2. When $\alpha = 1$ we get β-Laplace Transform.
3. When $\beta = e$ and $\alpha = 1$ we get standard Laplace Transform.

Definition 7. [20] Let $p(g)$ be a function defined for all $g \geq 0$ then the fractional β- Laplace-Carson Transform of order α which is denoted by ${}^{c}\mathcal{L}_{\beta}^{\alpha}\{p(g)\}$ and defined by

$$\begin{aligned} {}^{c}\mathcal{L}_{\beta}^{\alpha}\{p(g)\} = \mathfrak{F}_{\beta}^{\alpha}(g,\mu) &= \mu^{\alpha}\int_{0}^{\infty} \beta^{-(\mu g)^{\alpha}} p(g)(dg)^{\alpha}, 0<\beta\leq 1 \\ &= \int_{0}^{\infty} \mu^{\alpha} E_{\alpha}(-g^{\alpha}\mu^{\alpha}(\ln\beta)^{\alpha})p(g)(dg)^{\alpha} \end{aligned} \tag{10}$$

where $\mu \in \mathbb{C}$ and $E_{\alpha}(x)$ is Mittag- Leffler function $\sum_{n=0}^{\infty}\frac{x^n}{\alpha n!}$, $\beta \in (0,\infty)\backslash\{1\}$

Remark 2. We can see from the above definition (7),

1. When $\mu = 1, \beta = e$ becomes fractional Laplace Transforms.
2. When $\mu = 1, \alpha = 1$ becomes β- Laplace Transform.
3. When $\mu = 1, \alpha = 1$ and, $\beta = e$ becomes Laplace Transform.

Definition 8. If $p(g)$ is a function defined for all $g \geq 0$, the fractional β- Natural Transform of order α, denoted by $\mathcal{N}_{\beta}^{\alpha}\{p(g)\}$and defined by

$$\begin{aligned} \mathcal{N}_{\beta}^{\alpha}\{p(g)\} &= \left(\frac{1}{k}\right)^{\alpha}\int_{0}^{\infty} \beta^{\left(\left(\frac{-z}{k}\right)g\right)^{\alpha}} p(g)(dg)^{\alpha}, 0<\alpha\leq 1 \\ &= \left(\frac{1}{k}\right)^{\alpha}\int_{0}^{\infty} E_{\alpha}(-\left(\frac{z}{k}\right)^{\alpha} g^{\alpha}(\ln\beta)^{\alpha})p(g)(dg)^{\alpha} \end{aligned} \tag{11}$$

where $\mathcal{R}(z) > 0, k > 0, \beta \in (0,\infty)\backslash\{1\}$

Remark 3. From the above definition (8) we show that

1. If $\beta = e$ becomes fractional Natural Transforms.
2. If $\beta = e, \alpha = 1$ becomes Simple Natural Transform.
3. If $k = 1, \alpha = 1$ becomes β- Laplace Transform.
4. If $z = 1, \alpha = 1$ becomes β-Sumudu Transform.

4 | SUFFICIENT CONDITION FOR EXISTANCE OF β- LAPLACE TRANSFORM OF FRACTIONAL ORDER

Theorem 4. For any $(s\ln\beta)^{\alpha} > \gamma^{\alpha}$, modified Laplace transformations of fractional order exist if $q(t)$ is continuous in every finite interval in and piecewise the range $t \geq 0$ and is of exponential order *beta*.

proof: Since $q(t)$ is piecewise continuous, $\beta^{(-st)^{\alpha}}$ is integrable over any finite interval for $t \geq 0$

$$\therefore \quad \left|\int_0^\infty \beta^{-(st)^{\alpha}} q(t)(dt)^{\alpha}\right| \leq \int_0^\infty \left|\beta^{-(st)^{\alpha}} q(t)\right|(dt)^{\alpha}$$

$$< \int_0^\infty \beta^{-(st)^{\alpha}} M e^{(\gamma t)^{\alpha}} (dt)^{\alpha}$$

$$= M \int_0^\infty E_{\alpha}\left(-t^{\alpha} s^{\alpha} (\ln \beta)^{\alpha}\right) E_{\alpha}(\gamma^{\alpha} t^{\alpha})(dt)^{\alpha}$$

since q(t) is of exponential order β

$$= M \left[\frac{e^{-(s^{\alpha}(\ln \beta)^{\alpha} - \gamma^{\alpha})t^{\alpha}}}{-[(s \ln \beta)^{\alpha} - \gamma^{\alpha}]}\right]_0^\infty$$

$$= \frac{M}{(s \ln \beta)^{\alpha} - \gamma^{\alpha}}$$

$$\therefore \left|\mathcal{L}_{\beta}^{\alpha} q(t)\right| = \left|\int_0^\infty \beta^{-(st)^{\alpha}} q(t)(dt)^{\alpha}\right| < \frac{M}{(s \ln \beta)^{\alpha} - \gamma^{\alpha}}, \quad (s \ln \beta)^{\alpha} > \gamma^{\alpha}$$

4.1 | Some properties of Fractional β- Laplace Transform

Theorem 5. Let p, q be any arbitary constants and $f(x), g(x)$ are functions then

1. **Scaling Property:**
$$\mathcal{L}_{\beta}^{\alpha} \{f(t)\} = \mathfrak{F}_{\beta}^{\alpha}(s, pt)$$

2. **Linear Property:**
$$\mathcal{L}_{\beta}^{\alpha} \left\{p f_1(t) + q f_2(t)\right\} = p \mathcal{L}_{\beta}^{\alpha} \left\{f_1(t)\right\} + q \mathcal{L}_{\beta}^{\alpha} \left\{f_2(t)\right\}$$

3. **Shifting Property- I:** If $\mathcal{L}_{\beta}^{\alpha} \{f(t)\} = \mathfrak{F}_{\beta}^{\alpha}(s, t)$ then $\beta > 0 (\neq 1)$
$$\mathcal{L}_{\beta}^{\alpha} \left\{e^{(bt)^{\alpha}} f(t)\right\} = \mathfrak{F}_{\beta}^{\alpha} \left(s^{\alpha} \ln \beta - b^{\alpha}\right)$$

4. **Shifting Property- II:**
$$\mathcal{L}_{\beta}^{\alpha} \{g(p-b)u(p-b)\} = e^{-(sb)^{\alpha}} \mathcal{L}_{\beta}^{\alpha} \{g(p)\}$$

4.2 | Convolution Theorem for β- Laplace Fractional Order

Theorem 6. If we defined the convolution of order α of the two functions $n(x), s(x)$ by the equation

$$(n(x) \star s(x))^{\alpha} = \int_0^x n(x-u)s(u)(du)^{\alpha}$$

then

$$\mathcal{L}_{\beta}^{\alpha} \left[n(x) \star s(x)_{\alpha}\right] = \mathcal{L}_{\beta}^{\alpha} [n(x)] \, \mathcal{L}_{\beta}^{\alpha} [s(x)]$$

Proof: By the definition

$$\mathcal{L}_{\beta}^{\alpha} \left[n(x) \star s(x)_{\alpha}\right] = \int_0^\infty E_{\alpha}\left(-s^{\alpha} x^{\alpha} (\ln \beta)^{\alpha}\right) (dx)^{\alpha} \int_0^x n(x-u)s(u)(du)^{\alpha}$$

$$= \int_0^\infty E_{\alpha}\left(-s^{\alpha}(x-u)^{\alpha} (\ln \beta)^{\alpha}\right) E_{\alpha}\left(-s^{\alpha} u^{\alpha} (\ln \beta)^{\alpha}\right) (dx)^{\alpha} \int_0^x n(x-u)s(u)(du)^{\alpha}$$

This being a case, you make the change of variable $y = x - u$, $v = u$ to obtain

$$\mathcal{L}_{\beta}^{\alpha}\left[n(x) \star s(x)_{\alpha}\right] = \int_{0}^{\infty}\int_{0}^{\infty} E_{\alpha}\left(-s^{\alpha} y^{\alpha} (\ln \beta)^{\alpha}\right) E_{\alpha}\left(-s^{\alpha} v^{\alpha} (\ln \beta)^{\alpha}\right) (dy)^{\alpha} (dv)^{\alpha}$$

$$= \mathcal{L}_{\beta}^{\alpha}\left[n(x)\right] \mathcal{L}_{\beta}^{\alpha}\left[s(x)\right]$$

5 | β- LAPLACE TRANSFORM OF MITTAG-LEFFLER, RIEMANN LIOUVILLE'S INTEGRAL, CAPUTO FRACTIONAL DERIVATIVE AND ATANGANA-BALEANU DERIVATIVE

I. $\mathcal{L}_{\beta}\left\{f(t)\right\} = \int_{0}^{\infty} \beta^{-(st)} f(t)(dt) = \mathfrak{F}_{\beta}(s,t) = \int_{0}^{\infty} e^{-(st)\ln\beta} f(t) dt$
$\mathcal{L}_{\beta}\left\{t^{n}\right\} = \frac{n!}{s^{n+1}(\ln\beta)^{n+1}}$
Let $\mathcal{L}_{\beta}\left\{f(t)\right\} = \mathfrak{F}_{\beta}(s)$ and $\mathcal{L}_{\beta}\left\{g(t)\right\} = \mathfrak{G}_{\beta}(s)$ be such that $f(t)$and$g(t)$ are continuous functions on $[0,\infty)$ then their convolution $(f \star g)$ is defined by $\mathcal{L}_{\beta}\left\{(f \star g)(t)\right\} = \mathcal{L}_{\beta}\left[f(u)g(t-u)du\right] = \mathfrak{F}_{\beta}(s) \cdot \mathfrak{G}_{\beta}(s)$

II Mittag Leffler function is $E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\gamma)}$
$\mathcal{L}_{\beta}^{\alpha}\left[x^{\gamma-1}\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha})^{k}}{\Gamma(\alpha k+\gamma)}\right] = \int_{0}^{\infty} e^{-st(\ln\beta)} t^{\gamma-1} \sum_{k=0}^{\infty} \frac{\lambda^{k} t^{\alpha k}}{\Gamma(\alpha k+\gamma)} dt$
by simplification we get,

$$\mathcal{L}_{\beta}^{\alpha}\left[x^{\gamma-1}\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha})^{k}}{\Gamma(\alpha k+\gamma)}\right] = \frac{(s\ln\beta)^{\alpha-\gamma}}{(s\ln\beta)^{\alpha}-\lambda}$$

$$\therefore \quad \mathcal{L}_{\beta}^{\alpha}\left[x^{\gamma-1}E_{\alpha,\gamma}(\lambda x^{\alpha})\right] = \frac{(s\ln\beta)^{\alpha-\gamma}}{(s\ln\beta)^{\alpha}-\lambda}$$

If $\alpha = \gamma = \lambda = 1$ then $\mathcal{L}_{a}\left[E_{1,1}(x)\right] = \frac{1}{s\ln\beta-1} = \mathcal{L}_{\beta}(e^{x})$

If $\gamma = \lambda = 1$ then $\mathcal{L}_{\beta}^{\alpha}\left[E_{\alpha,1}(x^{\alpha})\right] = \frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^{\alpha}-1}$

III $\mathcal{L}_{\beta}^{\alpha}\left[\frac{d^{n}}{dx^{n}}\,{}_{0}I_{x}^{n-\alpha}f(x)\right] = (s\ln\beta)^{n}\mathcal{L}_{\beta}\left[{}_{0}I_{x}^{n-\alpha}f(x)\right] - \sum_{k=0}^{n-1}\frac{d^{n-k-1}}{dx^{n-k-1}}\,{}_{0}I_{x}^{n-\alpha}f(0)$

$$= (s\ln\beta)^{n}\left[(s\ln\beta)^{-(n-\alpha)}F(s)\right] - \sum_{k=0}^{n-1}(s\ln\beta)^{k}\frac{d^{n-k-1}}{dx^{n-k-1}}\,{}_{0}D_{x}^{\alpha-n}f(0)$$

$$\mathcal{L}_{\beta}^{\alpha}\left[f(x)\right] = (s\ln\beta)^{\alpha}F(s) - \sum_{k=0}^{n-1}(s\ln\beta)^{k}\,{}_{0}D_{x}^{\alpha-k-1}f(0)$$

IV $\mathcal{L}_{\beta}^{\alpha}\left[{}_{0}D_{x}^{\alpha}f(x)\right] = (s\ln\beta)^{\alpha}F(s) - \sum_{k=0}^{n-1}(s\ln\beta)^{\alpha-k-1}f^{(k)}(0)$
Let $f^{n}(x) = g(x)$
$\mathcal{L}_{\beta}^{\alpha}\left[{}_{0}I_{x}^{n-\alpha}g(x)\right] = (s\ln\beta)^{-(n-\alpha)}G(s), \left\{\because \mathcal{L}_{\beta}^{\alpha}\left[{}_{0}I_{x}^{\alpha}f(x)\right] = (s\ln\beta)^{-\alpha}F(s)\right\}$
where, $G(s) = \mathcal{L}_{\beta}\left[g(x)\right] = \mathcal{L}_{\beta}\left[f^{n}(x)\right]$
$= (s\ln\beta)^{n}F(s) - \sum_{k=0}^{n-1}(s\ln\beta)^{n-k-1}f^{(k)}(0)$

$$\mathcal{L}_{\beta}^{\alpha}\left[{}_{0}^{c}D_{x}^{\alpha}f(x)\right] = (s\ln\beta)^{\alpha-n}\left[(s\ln\beta)^{n}F(s) - \sum_{k=0}^{n-1}(s\ln\beta)^{n-k-1}f^{(k)}(0)\right]$$

$$= (s\ln\beta)^{\alpha}F(s) - \sum_{k=0}^{n-1}(s\ln\beta)^{\alpha-k-1}f^{(k)}(0)$$

V ${}_{0}I_{x}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}f(t)dt$

$\left\{\because (n \star m)(z) = \int_{0}^{z} n(z-t)m(t)dt \quad \text{and} \quad \mathcal{L}\left[n \star m\right] = N(s)M(s)\right\}$

$$\mathcal{L}_\beta\left[{}_0I_x^\alpha f(x)\right]=\mathcal{L}_\beta\left\{\frac{1}{\Gamma(\alpha)}\left[x^{\alpha-1}\star f(x)\right]\right\}$$

$$\mathcal{L}_\beta^\alpha\{f(x)\}={}_0I_x^\alpha f(x)=(s\ln\beta)^{-\alpha}F(s)$$

VI $\mathcal{L}_\beta^\alpha\left[{}_0^{RL}D_x^\alpha f(x)\right]=(s\ln\beta)^\alpha F(s)-\sum_{k=0}^{n-1}(s\ln\beta)^k{}_0D_x^{\alpha-k-1}f(x)$

$$\mathcal{L}_\beta\{f^n(x)\}=(s\ln\beta)^nF(s)-(s\ln\beta)^{n-1}f(0)-(s\ln\beta)^{n-2}f'(0)-\ldots-f^{n-1}(0)$$
$$\frac{d}{dx}E_\alpha(-\lambda x^\alpha)\leftrightarrow\frac{\lambda}{(s\ln\beta)^\alpha+\lambda}=-\left[(s\ln\beta)\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+\lambda}-1\right]$$
$$\frac{d}{dx}E_\alpha(-x^\alpha)\leftrightarrow\frac{1}{(s\ln\beta)^\alpha+1}=-\left[(s\ln\beta)\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+1}-1\right]$$

$$\frac{d^{-1}}{dx^{-1}}E_\alpha(-x^\alpha)\leftrightarrow\left[\frac{(s\ln\beta)^{\alpha-2}}{(s\ln\beta)^\alpha+1}\right]=\left[\frac{1}{(s\ln\beta)}\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+1}\right]$$
$$\frac{d^{-k}}{dx^{-k}}E_\alpha(-x^\alpha)\leftrightarrow\left[\frac{(s\ln\beta)^{\alpha-k-1}}{(s\ln\beta)^\alpha+1}\right]=\left[\frac{1}{(s\ln\beta)^k}\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+1}\right]$$

VII $\mathcal{L}_\beta^\alpha\left\{{}_0^{ABC}D_x^\alpha f(x)\right\}=\frac{k(\alpha)}{1-\alpha}\times\frac{(s\ln\beta)^\alpha}{(s\ln\beta)^\alpha+\frac{\alpha}{1-\alpha}}\left[F(s)-\frac{1}{(s\ln\beta)}f(0)\right]$

$$=\frac{k(\alpha)}{1-\alpha}\left[\frac{(s\ln\beta)^\alpha F(s)-(s\ln\beta)^{\alpha-1}f(0)}{(s\ln\beta)^\alpha+\frac{\alpha}{1-\alpha}}\right]$$

5.1 | Agarwal function for β- laplace transform of fractional order

In 1953, Agarwal generalised the Mittag Leffler function. Because of Agarwal's laplace transform, this function is particularly useful to fractional order system theory. The following is the definition of the function:

$$E_{\alpha,\omega}(x)=\sum_{m=0}^{\infty}\frac{x^{(m+\frac{\omega-1}{\alpha})}}{\Gamma(\alpha m+\omega)}$$
$$\mathcal{L}_\beta^\alpha\left[E_{\alpha,\omega}(x^\alpha)\right]=\frac{(s\ln\beta)^{\alpha-\omega}}{(s\ln\beta)^\alpha-1}$$

5.2 | β - Laplace transform of fractional order

The following expressions give some identities for β- laplace transform of fractional order of Mittag Leffler functions

$$\mathcal{L}_\beta^\alpha\left[x^{\omega-1}E_{\alpha,\omega}^k(\lambda x^\alpha)\right]=\frac{(s\ln\beta)^{\alpha-\omega}k!}{[(s\ln\beta)^\alpha-\lambda]^{k+1}}$$

Here, $E_{\alpha,\omega}^k=\frac{d^k}{dx^k}E_{\alpha,\omega}$
for $k>0$, the operation is differentiation of Mittag-Leffler functions and for $k<0$ the operation is integration of Mittag-Leffler functions.

If $\omega=1,k=0$ then,

1. $E_{\alpha,1}(\lambda x^\alpha)\to E_\alpha(\lambda x^\alpha)\leftrightarrow\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha-\lambda}$
2. $E_\alpha(-\lambda x^\alpha)\leftrightarrow\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+\lambda}$
3. $E_\alpha(-x^\alpha)\leftrightarrow\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^\alpha+1}$, $\lambda=1$

Table 1 :List of well-known function and their β-Fractional Laplace transform

Function	Time Expression $f(t)$	β- Fractional Laplace Transform $F(s)$
Mittag Leffler	$E_{\alpha}(\lambda x^{\alpha}) = \sum_{n=0}^{\infty} \frac{\lambda^{n} x^{n\alpha}}{\Gamma(n\alpha+1)}$	$\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)[(s\ln\beta)^{\alpha}-\lambda]}$
Agarwal	$E_{\alpha,\omega}(x^{\alpha}) = \sum_{m=0}^{\infty} \frac{x^{(m+\frac{\omega-1}{\alpha})\alpha}}{\Gamma(\alpha m+\omega)}$	$\frac{(s\ln\beta)^{\alpha-\omega}}{(s\ln\beta)^{\alpha}-1}$
Erdelyi	$E_{\alpha,\omega}(x) = \sum_{m=0}^{\infty} \frac{x^{m}}{\Gamma(\alpha m+\omega)}$	$\sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(\alpha m+\omega)(s\ln\beta)^{m+1}}$
Robotnov-Hartley	$F_{q}(\beta, x) \sum_{n=0}^{\infty} \frac{\beta^{n} x^{(}n+1)q-1}{\Gamma(n+1)q}$	$\frac{1}{(s\ln\beta)^{q}-q}$
Miller-Ross	$E_{x}(v, \beta) = \sum_{k=0}^{\infty} \frac{\beta^{k} x^{k+v}}{\Gamma(v+k+1)}$	$\frac{(s\ln\beta)^{-v}}{(s\ln\beta)-\beta}$

Table 2 :List of β- Laplace and β- inverse Laplace Transform of Fractional Order to Fractional Calculus

β- Laplace Transform of Fractional Order $F(s)$	Inverse $f(x)$
$\frac{(s\ln\beta)^{\alpha-1}}{(s\ln\beta)^{\alpha}\pm\lambda}, R(s) > \lvert\lambda\rvert^{1/\alpha}$	$E_{\alpha,1}(\pm\lambda x^{\alpha})$
$\frac{k!(s\ln a)^{\alpha-\omega}}{[(s\ln\beta)^{\alpha}\mp\lambda]^{k+1}}, R(s) > \lambda^{2}$	$x^{\alpha k+\omega-1} E^{k}_{\alpha,\omega}(\pm\lambda, x^{\alpha})$
$\frac{1}{(s\ln\beta)^{\alpha}}$	$\frac{x^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{\sqrt{(s\ln\beta)}}$	$\frac{1}{\sqrt{\pi x}}$

6 | APPLICATION

Example: $D^{\frac{1}{2}} f(t) + p f(t) = 0, \quad I^{\frac{1}{2}} f(t)|_{t=0} = C$

Solution: Applying the Modified Laplace Transform of order $\alpha = \frac{1}{2}$ We obtain

$$_{\beta}\mathcal{L}_{\frac{1}{2}} \left(D^{\frac{1}{2}} f(t) + p f(t) \right) = 0$$

$$(s\ln\beta)_{\beta}\mathcal{L}_{\frac{1}{2}} [f(t)(s)] - I^{\frac{1}{2}} f(t)|_{t=0} + p_{\beta}\mathcal{L}_{\frac{1}{2}} = 0$$

$$_{\beta}\mathcal{L}_{\frac{1}{2}} [f(t)(s)] = \frac{C}{(s\ln\beta) + p}$$

by applying the Modified Laplace inverse Transform of order $\alpha = \frac{1}{2}$

$$_{\beta}\mathcal{L}^{-1}_{\frac{1}{2}} \left[_{\beta}\mathcal{L}_{\frac{1}{2}} [f(t)(s)] \right] =_{\beta} \mathcal{L}^{-1}_{\frac{1}{2}} \left[\frac{C}{(s\ln\beta) + p} \right]$$

$$f(t) = C\, t^{\frac{-1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-p t^{\frac{1}{2}})$$

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Conflict of interest

The authors declare no potential conflict of interests.

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