CMMSE: On the generalized Fourier transform

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Abstract

In this paper, we introduce the theory of a generalized Fourier transform in order to solve differential equations with a generalized fractional derivative, and we state its main properties. In particular, we obtain the corresponding convolution, inverse and Plancherel formulas, and Hausdorff-Young inequality. We show that this generalized Fourier transform is useful in the study of fractional partial differential equations, by solving the fractional heat equation on the real line.

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Summary

In this paper, we introduce the theory of a generalized Fourier transform in order to solve differential equations with a generalized fractional derivative, and we state its main properties. In particular, we obtain the corresponding convolution, inverse and Plancherel formulas, and Hausdorff-Young inequality. We show that this generalized Fourier transform is useful in the study of fractional partial differential equations, by solving the fractional heat equation on the real line.

KEYWORDS:
fractional derivative; generalized Fourier transform

1 | INTRODUCTION

Differential and integral calculus provide numerous tools for solving modeled problems, but there are many phenomena whose formulation is much more precise if fractional calculus is used. The emergence of fractional calculus is as old as calculus and extends derivation and integration to arbitrary non-integer orders. *Liouville* to gave two definitions of derivative [1,2], treating the fractional order derivative as an integral, albeit with certain limitations. Anton *Karl Grünwald*, in 1867 [3], and *Aleksey Vasilievich Létnikov*, in 1868 [4], propose a new definition of fractional derivative based on the definition of iterated derivative, known as the Grünwald-Létnikov differo-integral operator. Later, in 1898, the definition given by *Liouville* was improved by *Riemann* in a posthumously published manuscript [5]. In 1969, *Michele Caputo* gave a new definition that allowed the physical interpretation of many problems, since it has ordinary initial conditions unlike the derivative of *Riemann*, so it is usually used in application problems [6].

The concept of conformable fractional derivative was introduced in [7], then [8,9,10,11,12,13] propose derivatives of local character, which opens a new horizon in fractional calculus.

Fractional calculus is now successfully used in a wide range of models in physics, economics and biology. Of particular importance are the physical applications in the theory of viscoelasticity, in the study of anomalous diffusion phenomena and electromagnetic theory. There is currently a growing interest in other very different fields such as circuit theory and the physics of the atmosphere. Also among economists, the use of fractional calculus concepts is consolidating. There are well-known fractional models such as that of the change of heat load intensity on the walls of a furnace, the *Bagley-Torvik* equation, the neural fractional order model, the deformation law or the model of spread of Dengue fever, where the advantage of using a non-integer formulation of the derivative is evident[14],[15],[16],[17],[18],[19],[20],[21],[22],[23],[24],[25],[26],[27],[28],[29],[30].

In the applications of fractional calculus we usually need to solve fractional differential equations. Hence, it is very useful to have the transform theory at our disposal. The papers[13] and[31] developed a theory of the Laplace transform for fractional differential equations. This theory was useful used in the study of fractional differential equations, see e.g.[30],[31],[32],[33].

In this paper, we introduce the theory of a generalized Fourier transform and we state its main properties. In particular, we obtain the corresponding convolution, inverse and Plancherel formulas, and Hausdorff-Young inequality. Plancherel formula allows to define this generalized Fourier transform in L^2 and, after that, Hausdorff-Young inequality allows to define this generalized Fourier transform in L^p for any $1 < p < 2$. A table of Fourier transforms is also included to facilitate the use of this theory. We show that this generalized Fourier transform is useful in the study of fractional partial differential equations, by solving the fractional heat equation on the real line:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial G_T^\alpha(\partial G_T^\alpha u)}{(\partial x^\alpha)^2}(x,t), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}. \end{cases}$$

where $G_T^\alpha u$ denotes the conformable fractional derivative operator of order $\alpha \in (0,1]$ on the variable x.

2 | PRELIMINARIES

Let us recall the definition of local generalized fractional derivative in[13].

Given $s \in \mathbb{R}$, we denote by $\lceil s \rceil$ the *upper integer part* of s, i.e., the smallest integer greater than or equal to s.

Definition 1. Given an interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{C}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $T(t,\alpha)$ on I for each α, the *derivative* $G_T^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G_T^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f\big(t - khT(t,\alpha)\big). \tag{1}$$

If $a = \min\{t \in I\}$ (respectively, $b = \max\{t \in I\}$), then $G_T^\alpha f(a)$ (respectively, $G_T^\alpha f(b)$) is defined with $h \to 0^-$ (respectively, $h \to 0^+$) instead of $h \to 0$ in the limit.

If we choose the function $T(t,\alpha)=t^{\lceil\alpha\rceil-\alpha}$, then we obtain the following particular case of G_T^α, defined in[7]. Note that $T(t,\alpha)=t^{\lceil\alpha\rceil-\alpha}=1$ for every $\alpha\in\mathbb{N}$.

Definition 2. Let I be an interval $I\subseteq(0,\infty)$, $f:I\to\mathbb{C}$ and $\alpha\in\mathbb{R}^+$. The *conformable derivative* $G^\alpha f$ of f of order α at the point $t\in I$ is defined by

$$G^\alpha f(t)=\lim_{h\to0}\frac{1}{h^{\lceil\alpha\rceil}}\sum_{k=0}^{\lceil\alpha\rceil}(-1)^k\binom{\lceil\alpha\rceil}{k}f\left(t-kht^{\lceil\alpha\rceil-\alpha}\right). \tag{2}$$

We know from the classical calculus that if f is a function defined in a neighborhood of the point t, and there exists $D^nf(t)$, then

$$D^nf(t)=\lim_{h\to0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(t-kh).$$

Therefore, if $\alpha=n\in\mathbb{N}$ and f is smooth enough, then Definition 2 coincides with the classical definition of the n-th derivative.

In[7] is defined a conformable derivative in the following way.

Definition 3. Given $f:(0,\infty)\to\mathbb{C}$ and $\alpha\in(0,1]$, the derivative of f of order α at the point t is defined by

$$T_\alpha f(t)=\lim_{h\to0}\frac{f(t)-f(t-ht^{1-\alpha})}{h}. \tag{3}$$

It is clear then that T_α is a particular case of G^α when $\alpha\in(0,1]$ and $T(t,\alpha)=t^{1-\alpha}$. See[34],[35] and[10] for more information on T_α.

The following results in[13] show some basic properties of the derivative G_T^α.

Lemma 1. Let I be an interval $I\subseteq\mathbb{R}$, $f:I\to\mathbb{C}$ and $\alpha\in\mathbb{R}^+$.

(1) If there exists $D^{\lceil\alpha\rceil}f$ at the point $t\in I$, then f is G_T^α-differentiable at t and $G_T^\alpha f(t)=T(t,\alpha)^{\lceil\alpha\rceil}D^{\lceil\alpha\rceil}f(t)$.

(2) If $\alpha\in(0,1]$, then f is G_T^α-differentiable at $t\in I$ if and only if f is differentiable at t; in this case, we have $G_T^\alpha f(t)=T(t,\alpha)f'(t)$.

Lemma 2. Let I be an interval $I\subseteq\mathbb{R}$, $f,g:I\to\mathbb{C}$ and $\alpha\in\mathbb{R}^+$. Assume that f,g are G_T^α-differentiable functions at $t\in I$. Then the following statements hold:

(1) $af+bg$ is G_T^α-differentiable at t for every $a,b\in\mathbb{R}$, and $G_T^\alpha(af+bg)(t)=a\,G_T^\alpha f(t)+b\,G_T^\alpha g(t)$.

(2) If $\alpha\in(0,1]$, then fg is G_T^α-differentiable at t and $G_T^\alpha(fg)(t)=f(t)G_T^\alpha g(t)+g(t)G_T^\alpha f(t)$.

(3) If $\alpha\in(0,1]$ and $g(t)\neq0$, then f/g is G_T^α-differentiable at t and $G_T^\alpha(\frac{f}{g})(t)=\frac{g(t)G_T^\alpha f(t)-f(t)G_T^\alpha g(t)}{g(t)^2}$.

(4) $G_T^\alpha(\lambda)=0$, for every $\lambda\in\mathbb{R}$.

(5) $G_T^\alpha(t^p)=\frac{\Gamma(p+1)}{\Gamma(p-\lceil\alpha\rceil+1)}t^{p-\lceil\alpha\rceil}T(t,\alpha)^{\lceil\alpha\rceil}$, for every $p\in\mathbb{R}\setminus\mathbb{Z}^-$.

(6) $G_T^\alpha(t^{-n})=(-1)^{\lceil\alpha\rceil}\frac{\Gamma(n+\lceil\alpha\rceil)}{\Gamma(n)}t^{-n-\lceil\alpha\rceil}T(t,\alpha)^{\lceil\alpha\rceil}$, for every $n\in\mathbb{Z}^+$.

Lemma 3. Let $\alpha\in(0,1]$, g a G_T^α-differentiable function at t and f a differentiable function at $g(t)$. Then $f\circ g$ is G_T^α-differentiable at t, and $G_T^\alpha(f\circ g)(t)=f'(g(t))\,G_T^\alpha g(t)$.

3 | ON THE GENERALIZED FOURIER TRANSFORM

In this section, we assume that the function $1/T$ is integrable on each compact interval in $\mathbb{R}$, and satisfies

$$\int_{-\infty}^{0} \frac{d\omega}{T(\omega,\alpha)} = \infty \qquad \text{and} \qquad \int_{0}^{\infty} \frac{d\omega}{T(\omega,\alpha)} = \infty$$

for each $0 < \alpha \leq 1$. We allow $T(t,\alpha)$ to be 0 on a set of zero Lebesgue measure.

Let us define for each $0 < \alpha \leq 1$, $t \in \mathbb{R}$ and $c \in \mathbb{C}$

$$E_\alpha(c,t) = \exp\Big(c \int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big).$$

This function has the following properties:

(1) $E_\alpha(c_1 + c_2, t) = E_\alpha(c_1,t) E_\alpha(c_2,t)$.

(2) $E_\alpha(c,t)$ is an eigenfunction for the operator G_T^α, since

$$\begin{aligned} G_T^\alpha(E_\alpha(c,t)) &= T(t,\alpha)\Big(\exp\Big(c \int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)\Big)' \\ &= T(t,\alpha)\, \exp\Big(c \int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big) \frac{c}{T(t,\alpha)} = c\, E_\alpha(c,t). \end{aligned}$$

Thus,

$$\big(E_\alpha(c,t)\big)' = c\, E_\alpha(c,t)\, \frac{1}{T(t,\alpha)}\,.$$

Given an integrable function $f : \mathbb{R} \to \mathbb{C}$, its Fourier transform is defined as

$$F[f](\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\, dt.$$

Given $0 < \alpha \leq 1$ and a measurable function $f : \mathbb{R} \to \mathbb{C}$ such that $f(t)/T(t,\alpha) \in L^1(\mathbb{R})$, we define its *generalized Fourier transform* as

$$\mathcal{F}_T^\alpha[f](\xi) = \int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\, f(t)\, \frac{dt}{T(t,\alpha)}$$

for any $\xi \in \mathbb{R}$.

The following properties of the generalized Fourier transform are elementary.

Proposition 1. Let $c \in \mathbb{R}$, $k_1, k_2 \in \mathbb{C}$, $0 < \alpha \leq 1$ and $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that $f(t)/T(t,\alpha), g(t)/T(t,\alpha) \in L^1(\mathbb{R})$. Then:

(1) There exists $\mathcal{F}_T^\alpha[\, k_1 f + k_2 g \,]$ and

$$\mathcal{F}_T^\alpha[\, k_1 f + k_2 g \,](\xi) = k_1 \mathcal{F}_T^\alpha[f](\xi) + k_2 \mathcal{F}_T^\alpha[g](\xi)$$

for any $\xi \in \mathbb{R}$.

(2) There exists $\mathcal{F}_T^\alpha\left[E_\alpha(ic,t)f(t)\right]$ and

$$\mathcal{F}_T^\alpha\left[E_\alpha(ic,t)f(t)\right](\xi) = \mathcal{F}_T^\alpha[f](\xi - c)$$

for any $\xi \in \mathbb{R}$.

Recall that a function $f : I \to \mathbb{C}$ is *absolutely continuous* on the compact interval I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_k (y_k - x_k) < \delta,$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon.$$

It is well-known that f is absolutely continuous on I if and only if there exists f' a.e. in I and $f(x) = f(a) + \int_a^x f'$ for every $a, x \in I$. If J is any interval in $\mathbb{R}$, we say that $f : J \to \mathbb{C}$ is *absolutely continuous* on J if it is absolutely continuous on each compact interval contained in J.

Let us consider the function

$$v(t) = \int_0^t \frac{d\omega}{T(\omega,\alpha)}.$$

Since $1/T(t,\alpha)$ is a positive function which is integrable on each compact interval in $\mathbb{R}$, we deduce that the function $v(t)$ is continuous and strictly increasing. Since

$$\lim_{t\to-\infty} v(t) = -\int_{-\infty}^0 \frac{d\omega}{T(\omega,\alpha)} = -\infty, \qquad \lim_{t\to\infty} v(t) = \int_0^\infty \frac{d\omega}{T(\omega,\alpha)} = \infty,$$

$v(t)$ is an homeomorphism on $\mathbb{R}$ and so is its inverse function, which we will denote by $w(x)$.

If $T(t,\alpha) = |t|^{1-\alpha}$, then

$$v(t) = \int_0^t \frac{d\omega}{T(\omega,\alpha)} = \int_0^t \omega^{\alpha-1}\, d\omega = \frac{1}{\alpha}\, t^\alpha$$

for $t \ge 0$, and $v(t) = -(-t)^\alpha/\alpha$ for $t < 0$, i.e., $v(t) = \operatorname{sgn}(t)|t|^\alpha/\alpha$, where $\operatorname{sgn}(t)$ is the function such that $\operatorname{sgn}(t) = 1$ if $t \ge 0$ and $\operatorname{sgn}(t) = -1$ otherwise. Thus, $w(x) = \operatorname{sgn}(x)(\alpha|x|)^{1/\alpha}$.

The following results summarize the main properties of the generalized Fourier transform.

Theorem 1. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f(t)/T(t,\alpha) \in L^1(\mathbb{R})$ for some $0 < \alpha \le 1$. Then $f \circ w \in L^1(\mathbb{R})$ and

$$\|f\circ w\|_1 = \|f/T\|_1, \qquad \mathcal{F}_T^\alpha[f](\xi) = F[f\circ w](\xi)$$

for any $\xi \in \mathbb{R}$, where F denotes the usual Fourier transform.

Proof. Note that

$$E_\alpha(-i\xi, t) = e^{-i\xi v(t)}.$$

Since $v(t)$ is an absolutely continuous function, the change of variable

$$x = v(t) = \int_0^t \frac{d\omega}{T(\omega,\alpha)}, \qquad dx = \frac{dt}{T(t,\alpha)}, \qquad t = w(x),$$

allows to obtain

$$\begin{aligned}\mathcal{F}_T^\alpha[f](\xi) &= \int_{-\infty}^{\infty} E_\alpha(-i\xi, t) f(t)\,\frac{dt}{T(t,\alpha)} = \int_{-\infty}^{\infty} e^{-i\xi v(t)} f(t)\,\frac{dt}{T(t,\alpha)} \\ &= \int_{-\infty}^{\infty} e^{-i\xi x} f(w(x))\,dx = F[f\circ w](\xi).\end{aligned}$$

This change of variable also gives

$$\|f\circ w\|_1 = \int_{-\infty}^{\infty} \left|f(w(x))\right| dx = \int_{-\infty}^{\infty} |f(t)|\,\frac{dt}{T(t,\alpha)} = \|f/T\|_1.$$

□

Corollary 1. Let $f : \mathbb{R} \to \mathbb{C}$ be a function in $L^1(\mathbb{R})$. Then $(f\circ v)/T \in L^1(\mathbb{R})$, there exists the generalized Fourier transform of $f\circ v$, and

$$\mathcal{F}_T^\alpha\big[f\circ v\big](\xi) = \mathcal{F}_T^\alpha\Big[f\Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\Big)\Big](\xi) = F[f](\xi)$$

for every $\xi \in \mathbb{R}$ and $0 < \alpha \le 1$.

Corollary 1 has the following consequence.

Proposition 2. There exists the generalized Fourier transform of the following functions for $0 < \alpha \le 1$:

(1) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\exp\left(-a\left|\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right|\right)\right](\xi) = \frac{2a}{\xi^2 + a^2}.$$

(2) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\frac{1}{\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)^2 + a^2}\right](\xi) = \frac{\pi}{a}\, e^{-a|\xi|}.$$

(3) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\exp\left(-a\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)^2\right)\right](\xi) = \sqrt{\frac{\pi}{a}}\,\exp\left(-\xi^2/(4a)\right).$$

(4) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\frac{1}{\sqrt{4\pi a}}\,\exp\left(-\frac{1}{4a}\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)^2\right)\right](\xi) = \exp\left(-a\xi^2\right).$$

(5) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\exp\left(-ia\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)^2\right)\right](\xi) = \sqrt{\frac{\pi}{a}}\; e^{-i\pi/4}\,\exp\left(i\xi^2/(4a)\right).$$

(6) If $a > 0$ and χ_A denotes the function with value 1 on the set A and 0 otherwise, then

$$\mathcal{F}_T^\alpha\left[\chi_{[-a,a]}\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)\right](\xi) = \frac{2\sin a\xi}{\xi}.$$

(7) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\frac{\sin\left(a\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)}{\int_0^t \frac{d\omega}{T(\omega,\alpha)}}\right](\xi) = \pi\,\chi_{[-a,a]}(\xi).$$

(8) If $a > 0$, then

$$\mathcal{F}_T^\alpha\left[\left(a - \left|\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right|\right)\chi_{[-a,a]}\left(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\right)\right](\xi) = \frac{\sin^2(a\xi/2)}{\xi^2}.$$

The following result shows that $\mathcal{F}_T^\alpha$ is the appropriate Fourier transform in order to work with the fractional derivative G_T^α.

Theorem 2. Let $0 < \alpha \le 1$ and $f : \mathbb{R} \to \mathbb{C}$ an absolutely continuous function such that $f/T, f' \in L^1(\mathbb{R})$. Then there exist $\mathcal{F}_T^\alpha[f]$ and $\mathcal{F}_T^\alpha[G_T^\alpha f]$, and

$$\mathcal{F}_T^\alpha[G_T^\alpha f](\xi) = i\xi\,\mathcal{F}_T^\alpha[f](\xi)$$

for every $\xi \in \mathbb{R}$.

Proof. Since $f/T, f' = G_T^\alpha f/T \in L^1(\mathbb{R})$, there exist $\mathcal{F}_T^\alpha[f]$ and $\mathcal{F}_T^\alpha[G_T^\alpha f]$.

Since f is an absolutely continuous function, if we apply integration by parts to the integral

$$\mathcal{F}_T^\alpha[G_T^\alpha f](\xi) = \int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\,G_T^\alpha f(t)\,\frac{dt}{T(t,\alpha)} = \int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\,f'(t)\,dt,$$

with

$$u = E_\alpha(-i\xi,t), \qquad du = -i\xi\,E_\alpha(-i\xi,t)\,\frac{dt}{T(t,\alpha)},$$

$$dv = f'(t)\,dt, \qquad v = f(t),$$

we obtain

$$\begin{aligned}\mathcal{F}_T^\alpha[G_T^\alpha f](\xi) &= \Big[E_\alpha(-i\xi,t)\,f(t)\Big]_{t=-\infty}^{t=\infty} + i\xi\int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\,f(t)\,\frac{dt}{T(t,\alpha)}\\ &= i\xi\,\mathcal{F}_T^\alpha[f](\xi),\end{aligned}$$

if we show that $\lim_{|t|\to\infty} f(t) = 0$.

Let us prove that $\lim_{t\to\infty} f(t) = 0$ (if $t \to -\infty$, the argument is similar):

Since $f' = G_T^\alpha f/T \in L^1(\mathbb{R})$ and $f(t) = f(0) + \int_0^t f'(s)\,ds$, there exists the limit $L = \lim_{t\to\infty} f(t) = f(0) + \int_0^\infty f'(s)\,ds$. Seeking for a contradiction assume that $L \neq 0$. Since $\int_0^\infty dt/T(t,\alpha) = \infty$, we have that $f/T \notin L^1([0,\infty))$, a contradiction. Thus, $\lim_{t\to\infty} f(t) = 0$ and the proof is finished. □

If we iterate this formula we obtain the following result. Define $G_T^{\alpha,1} f = G_T^\alpha f$ and $G_T^{\alpha,n} f = G_T^\alpha\big(G_T^{\alpha,n-1} f\big)$ for each $n \geq 2$.

Theorem 3. Let $0 < \alpha \leq 1$, $n \geq 1$ and $f : \mathbb{R} \to \mathbb{C}$ such that $f, G_T^\alpha f, G_T^{\alpha,2} f, \dots, G_T^{\alpha,n-1} f$ are absolutely continuous functions and $f/T, (G_T^\alpha f)/T, (G_T^{\alpha,2} f)/T, \dots, (G_T^{\alpha,n} f)/T \in L^1(\mathbb{R})$. Then there exist $\mathcal{F}_T^\alpha[f]$ and $\mathcal{F}_T^\alpha[G_T^{\alpha,n} f]$, and

$$\mathcal{F}_T^\alpha[G_T^{\alpha,n} f](\xi) = (i\xi)^n \mathcal{F}_T^\alpha[f](\xi)$$

for every $\xi \in \mathbb{R}$.

Theorem 4 below shows that the following integral operator plays an important role in our study.

$$J_T^\alpha(f)(t) = \int_0^t \frac{f(\omega)}{T(\omega,\alpha)}\,d\omega.$$

Theorem 4. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f/T, J_T^\alpha(f)/T \in L^1(\mathbb{R})$ for some $0 < \alpha \leq 1$. Then

$$\mathcal{F}_T^\alpha[J_T^\alpha(f)](\xi) = \frac{1}{i\xi}\,\mathcal{F}_T^\alpha[f](\xi)$$

for every $\xi \neq 0$.

Proof. Since $f/T \in L^1(\mathbb{R})$, we have that there exists $J_T^\alpha(f)(t)$ for every $t \in \mathbb{R}$. Also, $J_T^\alpha(f)(t)$ is absolutely continuous on $[0,\infty)$ and

$$(J_T^\alpha(f))'(t) = \frac{f(t)}{T(t,\alpha)}, \qquad G_T^\alpha(J_T^\alpha(f))(t) = f(t),$$

for almost every $t \in \mathbb{R}$. Thus, Theorem 2 applied to the function $J_T^\alpha(f)$ (since $J_T^\alpha(f)/T, f/T \in L^1(\mathbb{R})$) gives

$$\mathcal{F}_T^\alpha[f](\xi) = \mathcal{F}_T^\alpha[G_T^\alpha(J_T^\alpha(f))](\xi) = i\xi\,\mathcal{F}_T^\alpha[J_T^\alpha(f)](\xi).$$

□

Note that Theorem 4 shows that the integral operator J_T^α is the inverse of the generalized fractional derivative G_T^α.

Theorem 5 below allows to compute the derivatives of the generalized Fourier transform.

Theorem 5. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that

$$\frac{f(t)}{T(t,\alpha)},\ \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)}\Big)^n \frac{f(t)}{T(t,\alpha)} \in L^1(\mathbb{R}),$$

for some $0 < \alpha \leq 1$ and $n \geq 1$. Then:

$$\frac{d^k \mathcal{F}_T^\alpha[f]}{d\xi^k}(\xi) = (-i)^k \int_{-\infty}^{\infty} E_\alpha(-i\xi, t)\, f(t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^k \frac{dt}{T(t,\alpha)}$$

for every $\xi \in \mathbb{R}$ and $1 \leq k \leq n$.

Proof. Let us prove the formula by induction on $0 \leq k \leq n$, with the usual convention $d^0 g/d\xi^0 = g$. Since $f/T \in L^1(\mathbb{R})$, there exists $\mathcal{F}_T^\alpha[f]$. The formula trivially holds for $k = 0$. Consider $1 \leq k \leq n$ and assume that the induction hypothesis holds for $k-1$. Let us define

$$A = \Big\{ t \in \mathbb{R} : \Big| \int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big| \leq 1 \Big\}.$$

If $t \in A$, then

$$\Big| (-i)^k E_\alpha(-i\xi, t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^k \frac{f(t)}{T(t,\alpha)} \Big| \leq \frac{|f(t)|}{T(t,\alpha)}.$$

If $t \notin A$, then

$$\Big| (-i)^k E_\alpha(-i\xi, t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^k \frac{f(t)}{T(t,\alpha)} \Big| \leq \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^n \frac{|f(t)|}{T(t,\alpha)}.$$

Hence, we have for every $t \in \mathbb{R}$,

$$\Big| (-i)^k E_\alpha(-i\xi, t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^k \frac{f(t)}{T(t,\alpha)} \Big| \leq \frac{|f(t)|}{T(t,\alpha)} + \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^n \frac{|f(t)|}{T(t,\alpha)} \in L^1(\mathbb{R}).$$

Therefore, the induction hypothesis and dominated convergence theorem give

$$\begin{aligned}
\frac{d^k \mathcal{F}_T^\alpha[f]}{d\xi^k}(\xi) &= \frac{d}{d\xi} \Big(\int_{-\infty}^{\infty} (-i)^{k-1} E_\alpha(-i\xi, t)\, f(t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^{k-1} \frac{dt}{T(t,\alpha)} \Big) \\
&= (-i)^{k-1} \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi} \Big(E_\alpha(-i\xi, t) \Big) f(t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^{k-1} \frac{dt}{T(t,\alpha)} \\
&= (-i)^k \int_{-\infty}^{\infty} E_\alpha(-i\xi, t)\, f(t) \Big(\int_0^t \frac{d\omega}{T(\omega,\alpha)} \Big)^k \frac{dt}{T(t,\alpha)}.
\end{aligned}$$

□

Our next result shows that the generalized Fourier transform maps L^1 on L^∞:

Theorem 6. Let $f : \mathbb{R} \to \mathbb{C}$ be a function with $f/T \in L^1(\mathbb{R})$ for some $0 < \alpha \leq 1$. Then $\mathcal{F}_T^\alpha[f]$ is a bounded continuous function on $\mathbb{R}$ and

$$\left\| \mathcal{F}_T^\alpha[f] \right\|_\infty \leq \left\| f/T \right\|_1, \qquad \lim_{|\xi| \to \infty} \mathcal{F}_T^\alpha[f](\xi) = 0.$$

Proof. We have

$$\left|\mathcal{F}_T^{\alpha}[f](\xi)\right| = \Big|\int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\, f(t)\, \frac{dt}{T(t,\alpha)}\Big| \le \int_{-\infty}^{\infty} |f(t)|\, \frac{dt}{T(t,\alpha)} = \left\|f/T\right\|_1$$

for every $\xi \in \mathbb{R}$ and so,

$$\left\|\mathcal{F}_T^{\alpha}[f]\right\|_\infty \le \left\|f/T\right\|_1.$$

Also,

$$\begin{aligned}\left|\mathcal{F}_T^{\alpha}[f](\xi+h) - \mathcal{F}_T^{\alpha}[f](\xi)\right| &= \Big|\int_{-\infty}^{\infty} \big(E_\alpha(-i(\xi+h),t) - E_\alpha(-i\xi,t)\big)\, f(t)\, \frac{dt}{T(t,\alpha)}\Big| \\ &= \Big|\int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\big(E_\alpha(-ih,t)-1\big)\, f(t)\, \frac{dt}{T(t,\alpha)}\Big| \\ &\le \int_{-\infty}^{\infty} \Big|\big(E_\alpha(-ih,t)-1\big)\, f(t)\Big|\, \frac{dt}{T(t,\alpha)}\,.\end{aligned}$$

Since

$$\Big|\big(E_\alpha(-ih,t)-1\big)\, f(t)\Big|\, \frac{1}{T(t,\alpha)} \le 2\,\frac{|f(t)|}{T(t,\alpha)} \in L^1(\mathbb{R}),$$

and

$$\lim_{h\to 0} \big(E_\alpha(-ih,t)-1\big) = 0,$$

dominated convergence theorem gives

$$\lim_{h\to 0} \mathcal{F}_T^{\alpha}[f](\xi+h) = \mathcal{F}_T^{\alpha}[f](\xi).$$

Theorem 1 gives $\mathcal{F}_T^{\alpha}[f](\xi) = F[f\circ w](\xi)$ for every $\xi \in \mathbb{R}$. Since

$$\int_{-\infty}^{\infty} \left|f(w(x))\right| dx = \int_{-\infty}^{\infty} \left|f(t)\right| \frac{dt}{T(t,\alpha)} < \infty,$$

the classical Riemann-Lebesgue lemma gives

$$\lim_{|\xi|\to\infty} \mathcal{F}_T^{\alpha}[f](\xi) = \lim_{|\xi|\to\infty} F[f\circ w](\xi) = 0.$$

□

Let $f, g : \mathbb{R} \to \mathbb{C}$ be measurable functions and $0 < \alpha \le 1$. Let us define the *generalized convolution* of f and g as

$$(f * g)(t) = \int_{-\infty}^{\infty} f\Big(w\big(v(t) - v(\omega)\big)\Big)\, g(\omega)\, \frac{d\omega}{T(\omega,\alpha)}\,.$$

If $T(t,\alpha)=|t|^{1-\alpha}$, then $v(t)=\operatorname{sgn}(t)|t|^{\alpha}/\alpha$, $w(x)=\operatorname{sgn}(x)(\alpha|x|)^{1/\alpha}$, and we have in this case

$$
\begin{aligned}
(f*g)(t) &= \int_{-\infty}^{\infty} f\big(w\big(v(t)-v(\omega)\big)\big)\, g(\omega)\,\frac{d\omega}{T(\omega,\alpha)}\\
&= \int_{-\infty}^{\infty} f\Big(\operatorname{sgn}\big(\operatorname{sgn}(t)|t|^{\alpha}-\operatorname{sgn}(\omega)|\omega|^{\alpha}\big)\Big|\operatorname{sgn}(t)|t|^{\alpha}-\operatorname{sgn}(\omega)|\omega|^{\alpha}\Big|^{1/\alpha}\Big)\, g(\omega)\,|\omega|^{\alpha-1}\,d\omega.
\end{aligned}
$$

Lemma 4. The generalized convolution is a symmetric bilinear map. If $f,g:\mathbb{R}\to\mathbb{C}$ are functions with $f/T, g/T\in L^1(\mathbb{R})$ for some $0<\alpha\le 1$, then $(f*g)/T\in L^1(\mathbb{R})$ and

$$
\big\|(f*g)/T\big\|_1 \le \big\|f/T\big\|_1\big\|g/T\big\|_1 .
$$

Proof. We have

$$
\begin{aligned}
\big\|(f*g)/T\big\|_1 &= \Big|\int_{-\infty}^{\infty}(f*g)(t)\,\frac{dt}{T(t,\alpha)}\Big| \le \int_{-\infty}^{\infty}\big|(f*g)(t)\big|\,\frac{dt}{T(t,\alpha)}\\
&= \int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty} f\big(w\big(v(t)-v(\omega)\big)\big)\, g(\omega)\,\frac{d\omega}{T(\omega,\alpha)}\Big|\,\frac{dt}{T(t,\alpha)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|f\big(w\big(v(t)-v(\omega)\big)\big)\Big|\,\Big|g(\omega)\Big|\,v'(\omega)\,d\omega\; v'(t)\,dt\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|f\big(w\big(v(t)-v(\omega)\big)\big)\Big|\,v'(t)\,dt\,\Big|g(\omega)\Big|\,v'(\omega)\,d\omega.
\end{aligned}
$$

If we consider the change of variable in the integral on the variable t

$$
s=w\big(v(t)-v(\omega)\big),\quad v(s)=v(t)-v(\omega),\quad v'(s)\,ds=v'(t)\,dt,
$$

we obtain

$$
\begin{aligned}
\big\|(f*g)/T\big\|_1 &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|f\big(w\big(v(t)-v(\omega)\big)\big)\Big|\,v'(t)\,dt\,\Big|g(\omega)\Big|\,v'(\omega)\,d\omega\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|f(s)\Big|\,v'(s)\,ds\,\Big|g(\omega)\Big|\,v'(\omega)\,d\omega\\
&= \big\|f/T\big\|_1\big\|g/T\big\|_1 .
\end{aligned}
$$

The first statement is direct. □

The next result shows that the convolution is useful in the study of the generalized Fourier transform.

Theorem 7. If $f,g:\mathbb{R}\to\mathbb{C}$ are functions with $f/T, g/T\in L^1(\mathbb{R})$ for some $0<\alpha\le 1$, then there exists $\mathcal{F}_T^{\alpha}[f*g]$ and

$$
\mathcal{F}_T^{\alpha}[f*g](\xi)=\mathcal{F}_T^{\alpha}[f](\xi)\,\mathcal{F}_T^{\alpha}[g](\xi)
$$

for every $\xi\in\mathbb{R}$.

Proof. Lemma 4 gives that $(f * g)/T \in L^1(\mathbb{R})$ and so, there exists $\mathcal{F}_T^\alpha[f * g]$. The argument in the proof of Lemma 4 gives that we can apply Fubini's Theorem in order to obtain

$$\begin{aligned}
\mathcal{F}_T^\alpha[f * g](\xi) &= \int_{-\infty}^{\infty} E(-i\xi, t)(f * g)(t)\,\frac{dt}{T(t,\alpha)} = \int_{-\infty}^{\infty} e^{-i\xi v(t)}(f * g)(t)\,v'(t)\,dt \\
&= \int_{-\infty}^{\infty} e^{-i\xi v(t)} \int_{-\infty}^{\infty} f\big(w\big(v(t)-v(\omega)\big)\big)\,g(\omega)\,v'(\omega)\,d\omega\; v'(t)\,dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(v(t)-v(\omega))} f\big(w\big(v(t)-v(\omega)\big)\big)\,v'(t)\,dt\; e^{-i\xi v(\omega)} g(\omega)\,v'(\omega)\,d\omega.
\end{aligned}$$

If we consider the change of variable in the integral on the variable t

$$s = w\big(v(t)-v(\omega)\big), \quad v(s) = v(t)-v(\omega), \quad v'(s)\,ds = v'(t)\,dt,$$

we obtain

$$\begin{aligned}
\mathcal{F}_T^\alpha[f * g](\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(v(t)-v(\omega))} f\big(w\big(v(t)-v(\omega)\big)\big)\,v'(t)\,dt\, e^{-i\xi v(\omega)} g(\omega)\,v'(\omega)\,d\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi v(s)} f(s)\,v'(s)\,ds\, e^{-i\xi v(\omega)} g(\omega)\,v'(\omega)\,d\omega \\
&= \mathcal{F}_T^\alpha[f](\xi)\,\mathcal{F}_T^\alpha[g](\xi)
\end{aligned}$$

for every $\xi \in \mathbb{R}$. □

In what follows we denote by μ the measure on $\mathbb{R}$ defined by

$$d\mu(t) = v'(t)\,dt = \frac{dt}{T(t,\alpha)}\,.$$

The next result shows that the generalized convolution is defined if f and g have minimal integrability properties, generalizing Lemma 4.

Theorem 8. If $1 \le p, q, r \le \infty$ with $1/r = 1/p + 1/q - 1$ and $f \in L^p(\mathbb{R},\mu)$, $g \in L^q(\mathbb{R},\mu)$ for some $0 < \alpha \le 1$, then $f * g \in L^r(\mathbb{R},\mu)$ and

$$\big\|f * g\big\|_{L^r(\mathbb{R},\mu)} \le \big\|f\big\|_{L^p(\mathbb{R},\mu)} \big\|g\big\|_{L^q(\mathbb{R},\mu)}.$$

Proof. Fix $1 \le p < \infty$ and $f \in L^p(\mathbb{R},\mu)$, and consider the linear map $Mg = f * g$. If p' is the dual exponent of p, i.e., $1/p + 1/p' = 1$, we are going to prove that $M : L^{p'}(\mathbb{R},\mu) \to L^\infty(\mathbb{R},\mu)$ is a continuous linear map. Hölder inequality gives

$$\begin{aligned}
\big|(f * g)(t)\big| &= \Big|\int_{-\infty}^{\infty} f\big(w\big(v(t)-v(\omega)\big)\big)\,g(\omega)\,v'(\omega)\,d\omega\Big| \\
&\le \Big(\int_{-\infty}^{\infty} \Big|f\big(w\big(v(t)-v(\omega)\big)\big)\Big|^p v'(\omega)\,d\omega\Big)^{1/p} \Big(\int_{-\infty}^{\infty} \big|g(\omega)\big|^{p'} v'(\omega)\,d\omega\Big)^{1/p'}.
\end{aligned}$$

The change of variable

$$s = w\big(v(t) - v(\omega)\big), \quad v(s) = v(t) - v(\omega), \quad v'(s)\,ds = -v'(\omega)\,d\omega,$$

gives

$$\begin{aligned}
\Big|(f * g)(t)\Big| &\le \Big(\int_{-\infty}^{\infty} \Big|f\Big(w\big(v(t) - v(\omega)\big)\Big)\Big|^p v'(\omega)\,d\omega\Big)^{1/p}\Big(\int_{-\infty}^{\infty} \Big|g(\omega)\Big|^{p'} v'(\omega)\,d\omega\Big)^{1/p'} \\
&\le \Big(\int_{-\infty}^{\infty} \Big|f(s)\Big|^p v'(s)\,ds\Big)^{1/p}\Big(\int_{-\infty}^{\infty} \Big|g(\omega)\Big|^{p'} v'(\omega)\,d\omega\Big)^{1/p'} \\
&= \Big\|f\Big\|_{L^p(\mathbb{R},\mu)}\Big\|g\Big\|_{L^{p'}(\mathbb{R},\mu)},
\end{aligned}$$

for every $t \in \mathbb{R}$. Hence,

$$\Big\|f * g\Big\|_{L^\infty(\mathbb{R},\mu)} = \Big\|f * g\Big\|_{L^\infty(\mathbb{R})} \le \Big\|f\Big\|_{L^p(\mathbb{R},\mu)}\Big\|g\Big\|_{L^{p'}(\mathbb{R},\mu)}.$$

If $p = \infty$, then the inequality is direct.

Let us prove now that $M : L^1(\mathbb{R},\mu) \to L^p(\mathbb{R},\mu)$ is a continuous linear map. By applying Minkowski integral inequality we obtain

$$\begin{aligned}
\Big\|f * g\Big\|_{L^p(\mathbb{R},\mu)} &= \Big(\int_{-\infty}^{\infty} \Big|(f * g)(t)\Big|^p v'(t)\,dt\Big)^{1/p} \\
&= \Big(\int_{-\infty}^{\infty} \Big|\int_{-\infty}^{\infty} f\Big(w\big(v(t) - v(\omega)\big)\Big)\, g(\omega)\, v'(\omega)\,d\omega\Big|^p v'(t)\,dt\Big)^{1/p} \\
&\le \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \Big|f\Big(w\big(v(t) - v(\omega)\big)\Big)\Big|^p v'(t)\,dt\Big)^{1/p} |g(\omega)|\, v'(\omega)\,d\omega \\
&= \Big\|f\Big\|_{L^p(\mathbb{R},\mu)}\Big\|g\Big\|_{L^1(\mathbb{R},\mu)}.
\end{aligned}$$

Riesz-Thorin Interpolation Theorem gives that if

$$\begin{aligned}
\frac{1}{p_\theta} &= \frac{1-\theta}{1} + \frac{\theta}{p'} = 1 - \theta + \theta\Big(1 - \frac{1}{p}\Big) = 1 - \frac{\theta}{p}, \\
\frac{1}{q_\theta} &= \frac{1-\theta}{p} + \frac{\theta}{\infty} = \frac{1-\theta}{p},
\end{aligned}$$

for $\theta \in [0,1]$, then $M : L^{p_\theta}(\mathbb{R},\mu) \to L^{q_\theta}(\mathbb{R},\mu)$ is a continuous linear map and

$$\Big\|f * g\Big\|_{L^{q_\theta}(\mathbb{R},\mu)} \le \Big\|f\Big\|_{L^p(\mathbb{R},\mu)}\Big\|g\Big\|_{L^{p_\theta}(\mathbb{R},\mu)}.$$

If we define $q = p_\theta$ and $r = q_\theta$, then the result follows from

$$\frac{1}{q} = 1 - \frac{\theta}{p}, \qquad \frac{1}{r} = \frac{1-\theta}{p}, \qquad \frac{1}{p} + \frac{1}{q} - 1 = \frac{1-\theta}{p} = \frac{1}{r}.$$

□

4 | THE INVERSE FORMULA AND THE DEFINITION IN L^2

The following is a version of Fourier inversion theorem.

Theorem 9. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f/T, \mathcal{F}_T^\alpha[f] \in L^1(\mathbb{R})$ for some $0 < \alpha \le 1$. Then for almost every t

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_\alpha(i\xi, t) \mathcal{F}_T^\alpha[f](\xi)\, d\xi.$$

Furthermore, the equality holds for every t such that f is continuous at t.

Proof. Theorem 1 gives $\mathcal{F}_T^\alpha[f](\xi) = F[f \circ w](\xi)$, for any $\xi \in \mathbb{R}$, where F denotes the usual Fourier transform. If $s = v(t)$, then $t = w(s)$ and we have for almost every s (and so, for almost every t)

$$\begin{aligned} f(t) = (f \circ w)(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi s} F[f \circ w](\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi v(t)} \mathcal{F}_T^\alpha[f](\xi)\, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_\alpha(i\xi, t) \mathcal{F}_T^\alpha[f](\xi)\, d\xi. \end{aligned}$$

Also, the equality holds for every s such that f is continuous at $w(s)$. □

Proposition 3. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f/T \in L^1(\mathbb{R})$ for some $0 < \alpha \le 1$. Then $\mathcal{F}_T^\alpha[\overline{f}\,](\xi) = \overline{\mathcal{F}_T^\alpha[f](-\xi)}$ for every $\xi \in \mathbb{R}$.

Proof. We have

$$\overline{\mathcal{F}_T^\alpha[f](-\xi)} = \overline{\int_{-\infty}^{\infty} e^{i\xi v(t)} f(t) \frac{dt}{T(t,\alpha)}} = \int_{-\infty}^{\infty} e^{-i\xi v(t)} \overline{f(t)} \frac{dt}{T(t,\alpha)} = \mathcal{F}_T^\alpha[\overline{f}\,](\xi),$$

for every $\xi \in \mathbb{R}$. □

Note that, in particular, $\left|\mathcal{F}_T^\alpha[\overline{f}\,](\xi)\right| = \left|\mathcal{F}_T^\alpha[f](-\xi)\right|$ for every $\xi \in \mathbb{R}$.

Next, we prove a main result.

Theorem 10. Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that $f/T, g/T, \mathcal{F}_T^\alpha[g] \in L^1(\mathbb{R})$ for some $0 < \alpha \le 1$. Then $f\overline{g}/T, \mathcal{F}_T^\alpha[f]\overline{\mathcal{F}_T^\alpha[g]} \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} f(t) \overline{g(t)} \frac{dt}{T(t,\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_T^\alpha[f](\xi)\, \overline{\mathcal{F}_T^\alpha[g](\xi)}\, d\xi.$$

Proof. Since $f/T, g/T, \mathcal{F}_T^\alpha[g] \in L^1(\mathbb{R})$ we can apply Theorem 9 and Fubini's theorem in the following integrals:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,\frac{dt}{T(t,\alpha)} &= \int_{-\infty}^{\infty} f(t)\overline{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} E_\alpha(i\xi,t)\mathcal{F}_T^\alpha[g](\xi)\,d\xi\right)}\frac{dt}{T(t,\alpha)} \\
&= \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} E_\alpha(-i\xi,t)\overline{\mathcal{F}_T^\alpha[g](\xi)}\,d\xi\right)\frac{dt}{T(t,\alpha)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathcal{F}_T^\alpha[g](\xi)}\left(\int_{-\infty}^{\infty} f(t)E_\alpha(-i\xi,t)\,\frac{dt}{T(t,\alpha)}\right)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}_T^\alpha[f](\xi)\,\overline{\mathcal{F}_T^\alpha[g](\xi)}\,d\xi,
\end{aligned}$$

and so, Fubini's theorem gives $f\overline{g}/T, \mathcal{F}_T^\alpha[f]\overline{\mathcal{F}_T^\alpha[g]} \in L^1(\mathbb{R})$. □

Theorem 10 has the following consequence, which is a weak version of Plancherel theorem.

Theorem 11. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f/T, \mathcal{F}_T^\alpha[f] \in L^1(\mathbb{R})$ for some $0 < \alpha \le 1$. Then $f^2/T, \mathcal{F}_T^\alpha[f]^2 \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(t)|^2\,\frac{dt}{T(t,\alpha)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}_T^\alpha[f](\xi)\right|^2 d\xi.$$

Let us define

$$K_a(t) = \frac{1}{\sqrt{4\pi a}}\,\exp\left(-\frac{1}{4a}v(t)^2\right).$$

Proposition 2 gives

$$\mathcal{F}_T^\alpha\left[K_a\right](\xi) = \exp\left(-a\xi^2\right).$$

We will use several times the following result.

Theorem 12. If $1 \le p < \infty$, $0 < \alpha \le 1$ and $f \in L^p(\mathbb{R},\mu)$, then $K_a * f \in L^p(\mathbb{R},\mu)$ and

$$\lim_{a\to 0^+}\left\|(K_a * f) - f\right\|_{L^p(\mathbb{R},\mu)}^p = \lim_{a\to 0^+}\int_{-\infty}^{\infty}\left|(K_a * f)(t) - f(t)\right|^p\frac{dt}{T(t,\alpha)} = 0.$$

Furthermore, if f is continuous at t, then $\lim_{a\to 0^+}(K_a * f)(t) = f(t)$.

Proof. If we consider the change of variable

$$s = \frac{1}{\sqrt{4a}}\big(v(t) - v(\omega)\big) \qquad ds = \frac{-1}{\sqrt{4a}}\,v'(\omega)\,d\omega,$$

we obtain

$$(K_a * 1)(t) = \int_{-\infty}^{\infty} K_a\big(w\big(v(t) - v(\omega)\big)\big)\, \frac{d\omega}{T(\omega,\alpha)}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a}}\, \exp\Big(-\frac{1}{4a}\big(v(t) - v(\omega)\big)^2\Big)\, v'(\omega)\, d\omega$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, \exp\big(-s^2\big)\, ds = 1.$$

Minkowski integral inequality gives

$$\Big\|(K_a * f) - f\Big\|_{L^p(\mathbb{R},\mu)} = \Big(\int_{-\infty}^{\infty} \Big|(K_a * f)(t) - f(t)(K_a * 1)(t)\Big|^p v'(t)\, dt\Big)^{1/p}$$
$$= \Big(\int_{-\infty}^{\infty} \Big|\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a}}\, \exp\Big(-\frac{1}{4a}\big(v(t) - v(\omega)\big)^2\Big)\big(f(\omega) - f(t)\big)\, v'(\omega)\, d\omega\Big|^p v'(t)\, dt\Big)^{1/p}$$
$$\le \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \frac{1}{(4\pi a)^{p/2}}\, \exp\Big(-\frac{p}{4a}\big(v(t) - v(\omega)\big)^2\Big)\Big|f(\omega) - f(t)\Big|^p v'(t)\, dt\Big)^{1/p} v'(\omega)\, d\omega$$
$$= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \frac{1}{(4\pi a)^{p/2}}\, \exp\Big(-\frac{p}{4a}\big(x - y\big)^2\Big)\Big|f(w(x)) - f(w(y))\Big|^p dx\Big)^{1/p} dy$$
$$= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \Big|(f\circ w)(x) - (f\circ w)(x - s)\Big|^p dx\Big)^{1/p} \frac{1}{\sqrt{4\pi a}}\, \exp\Big(-\frac{1}{4a}s^2\Big)\, ds$$

It is well known that the L^p-modulus of continuity satisfies

$$\lim_{s\to 0}\Big(\int_{-\infty}^{\infty} \Big|(f\circ w)(x) - (f\circ w)(x - s)\Big|^p dx\Big)^{1/p} = 0,$$

since $f \in L^p(\mathbb{R},\mu)$ implies $f\circ w \in L^p(\mathbb{R})$. Fix $\varepsilon > 0$ and choose $\delta > 0$ with

$$\Big(\int_{-\infty}^{\infty} \Big|(f\circ w)(x) - (f\circ w)(x - s)\Big|^p dx\Big)^{1/p} < \frac{\varepsilon}{2}$$

for every s with $|s| < \delta$. Also, we have

$$\lim_{a\to 0^+} \int_{|s|\ge\delta} \frac{1}{\sqrt{4\pi a}}\, \exp\Big(-\frac{1}{4a}s^2\Big)\, ds = \lim_{a\to 0^+} \int_{|t|\ge\delta/\sqrt{a}} \frac{1}{\sqrt{4\pi}}\, \exp\Big(-\frac{1}{4}t^2\Big)\, dt = 0.$$

Choose $a > 0$ with

$$\int_{|s|\ge\delta} \frac{1}{\sqrt{4\pi a}}\, \exp\Big(-\frac{1}{4a}s^2\Big)\, ds < \frac{\varepsilon}{4\|f\circ w\|_{L^p(\mathbb{R})}}\,.$$

Therefore,

$$\begin{aligned}
&\int\limits_{|s|<\delta}\Big(\int\limits_{-\infty}^{\infty}\Big|(f\circ w)(x)-(f\circ w)(x-s)\Big|^p\,dx\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&\quad<\int\limits_{|s|<\delta}\frac{\varepsilon}{2}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&\quad<\frac{\varepsilon}{2}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds=\frac{\varepsilon}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{|s|\geq\delta}\Big(\int\limits_{-\infty}^{\infty}\Big|(f\circ w)(x)-(f\circ w)(x-s)\Big|^p\,dx\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&\leq\int\limits_{|s|\geq\delta}\Big(\int\limits_{-\infty}^{\infty}\Big(\Big|(f\circ w)(x)\Big|+\Big|(f\circ w)(x-s)\Big|\Big)^p\,dx\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&\leq\int\limits_{|s|\geq\delta}\Big(\int\limits_{-\infty}^{\infty}\Big(2^{p-1}\Big|(f\circ w)(x)\Big|^p+2^{p-1}\Big|(f\circ w)(x-s)\Big|^p\Big)\,dx\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&=\int\limits_{|s|\geq\delta}\Big(2^{p-1}\|f\circ w\|^p_{L^p(\mathbb{R})}+2^{p-1}\|f\circ w\|_{L^p(\mathbb{R})}\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&=2\,\|f\circ w\|_{L^p(\mathbb{R})}\int\limits_{|s|\geq\delta}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds\\
&<2\,\|f\circ w\|_{L^p(\mathbb{R})}\frac{\varepsilon}{4\|f\circ w\|_{L^p(\mathbb{R})}}=\frac{\varepsilon}{2}.
\end{aligned}$$

Hence, we conclude

$$\int\limits_{-\infty}^{\infty}\Big(\int\limits_{-\infty}^{\infty}\Big|(f\circ w)(x)-(f\circ w)(x-s)\Big|^p\,dx\Big)^{1/p}\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}s^2\Big)\,ds<\varepsilon,$$

and this finishes the proof of the first statement.

The second statement can be proved with an argument similar to the previous one. ☐

Theorems 11 and 12 allow to prove the next version of Plancherel theorem.

Theorem 13. Let $f:\mathbb{R}\to\mathbb{C}$ be a function such that $f\in L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ for some $0<\alpha\leq 1$. Then $\mathcal{F}_T^\alpha[f]\in L^2(\mathbb{R})$ and

$$\int\limits_{-\infty}^{\infty}|f(t)|^2\frac{dt}{T(t,\alpha)}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big|\mathcal{F}_T^\alpha[f](\xi)\Big|^2\,d\xi.$$

Proof. Recall that

$$K_a(t)=\frac{1}{\sqrt{4\pi a}}\exp\Big(-\frac{1}{4a}v(t)^2\Big),\qquad \mathcal{F}_T^\alpha\left[K_a\right](\xi)=\exp\big(-a\xi^2\big).$$

For each $f\in L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ let us consider the function K_a*f. Theorem 12 gives that $K_a*f\in L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ and Theorem 7 implies that its generalized Fourier transform satisfies $\mathcal{F}_T^\alpha\left[K_a*f\right]=\mathcal{F}_T^\alpha\left[K_a\right]\mathcal{F}_T^\alpha\left[f\right]$. Since $\mathcal{F}_T^\alpha\left[K_a\right]\in L^1(\mathbb{R})$

and $\mathcal{F}_T^\alpha[f] \in L^\infty(\mathbb{R})$ by Theorem 6, we have $\mathcal{F}_T^\alpha\left[K_a * f\right] \in L^1(\mathbb{R})$ and so, Theorem 11 gives

$$\int_{-\infty}^{\infty} \left|(K_a * f)(t)\right|^2 \frac{dt}{T(t,\alpha)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}_T^\alpha[K_a * f](\xi)\right|^2 d\xi$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-2a\xi^2\right)\left|\mathcal{F}_T^\alpha[f](\xi)\right|^2 d\xi.$$

Since $f \in L^2(\mathbb{R},\mu)$, Theorem 12 gives

$$\lim_{a\to 0^+}\int_{-\infty}^{\infty} \left|(K_a * f)(t)\right|^2 \frac{dt}{T(t,\alpha)} = \int_{-\infty}^{\infty} |f(t)|^2 \frac{dt}{T(t,\alpha)}.$$

Since $\exp(-2a\xi^2)$ increases to 1 as $a \to 0^+$, monotonous convergence theorem gives

$$\lim_{a\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-2a\xi^2\right)\left|\mathcal{F}_T^\alpha[f](\xi)\right|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}_T^\alpha[f](\xi)\right|^2 d\xi.$$

Hence,

$$\int_{-\infty}^{\infty} |f(t)|^2 \frac{dt}{T(t,\alpha)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}_T^\alpha[f](\xi)\right|^2 d\xi.$$

Since $f \in L^2(\mathbb{R},\mu)$, we conclude that $\mathcal{F}_T^\alpha[f] \in L^2(\mathbb{R})$. □

Since every simple function is contained in $L^1(\mathbb{R},\mu) \cap L^2(\mathbb{R},\mu)$ and the set of simple functions is dense in $L^2(\mathbb{R},\mu)$, we conclude that $L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ is dense in $L^2(\mathbb{R},\mu)$. Hence, given $f \in L^2(\mathbb{R},\mu)$, there exists a sequence $\{f_n\} \subset L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ with

$$\lim_{n\to\infty} \|f - f_n\|_{L^2(\mathbb{R},\mu)} = 0.$$

Since each f_n belongs to $L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$, Theorem 13 gives

$$\|f_n\|_{L^2(\mathbb{R},\mu)} = \frac{1}{\sqrt{2\pi}}\left\|\mathcal{F}_T^\alpha[f_n]\right\|_{L^2(\mathbb{R})}$$

for every n. Since $\lim_{n\to\infty}\|f_n - f\|_{L^2(\mathbb{R},\mu)} = 0$, $\{f_n\}$ is a Cauchy sequence and, as

$$\left\|\mathcal{F}_T^\alpha[f_n] - \mathcal{F}_T^\alpha[f_m]\right\|_{L^2(\mathbb{R})} = \left\|\mathcal{F}_T^\alpha[f_n - f_m]\right\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\|f_n - f_m\|_{L^2(\mathbb{R},\mu)},$$

$\{\mathcal{F}_T^\alpha[f_n]\}$ is also a Cauchy sequence in $L^2(\mathbb{R})$ and, since $L^2(\mathbb{R})$ is complete, $\{\mathcal{F}_T^\alpha[f_n]\}$ converges to a function $g \in L^2(\mathbb{R})$. We define $\mathcal{F}_T^\alpha[f] = g$.

In order to check that $\mathcal{F}_T^\alpha[f]$ is well defined, let us choose another sequence $\{f_n^*\} \subset L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ such that $\lim_{n\to\infty}\|f_n^* - f\|_{L^2(\mathbb{R},\mu)} = 0$. We know that $\{\mathcal{F}_T^\alpha[f_n^*]\}$ converges to a function $g^* \in L^2(\mathbb{R})$. We need to check that $g^* = g$:

Let us consider the sequence $\{f_n^{**}\} \subset L^1(\mathbb{R},\mu)\cap L^2(\mathbb{R},\mu)$ which is the union of $\{f_n\}$ and $\{f_n^*\}$. Thus, $\lim_{n\to\infty}\|f_n^{**} - f\|_{L^2(\mathbb{R},\mu)} = 0$ and $\{\mathcal{F}_T^\alpha[f_n^{**}]\}$ converges to a function $g^{**} \in L^2(\mathbb{R})$. Since $\{f_n\}$ (respectively, $\{f_n^*\}$) is a subsequence of $\{f_n^{**}\}$, we have $g = g^{**}$ (respectively, $g^* = g^{**}$). Hence, $g^* = g$ and the generalized Fourier transform is well defined in the whole $L^2(\mathbb{R}, \mu)$.

The following result summarizes the properties of the generalized Fourier transform in $L^2(\mathbb{R}, \mu)$.

Theorem 14. The previous definition associates to each $f \in L^2(\mathbb{R}, \mu)$ a function $\mathcal{F}_T^\alpha[f] \in L^2(\mathbb{R})$ with the following properties:

(1) If $f \in L^1(\mathbb{R}, \mu) \cap L^2(\mathbb{R}, \mu)$, then $\mathcal{F}_T^\alpha[f]$ is the previously defined generalized Fourier transform.

(2) Plancherel theorem holds for every $f \in L^2(\mathbb{R}, \mu)$:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \frac{dt}{T(t,\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \mathcal{F}_T^\alpha[f](\xi) \right|^2 d\xi.$$

(3) The mapping $f \mapsto \mathcal{F}_T^\alpha[f]$ is a Hilbert's space isomorphism from $L^2(\mathbb{R}, \mu)$ onto $L^2(\mathbb{R})$: $\langle f, g\rangle = \frac{1}{2\pi} \langle \mathcal{F}_T^\alpha[f], \mathcal{F}_T^\alpha[g]\rangle$, i.e.,

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)} \, \frac{dt}{T(t,\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_T^\alpha[f](\xi) \, \overline{\mathcal{F}_T^\alpha[g](\xi)} \, d\xi \, ,$$

for every $f, g \in L^2(\mathbb{R}, \mu)$.

(4) If we define

$$U_R(\xi) = \int_{-R}^{R} E_\alpha(-i\xi, t) \, f(t) \, \frac{dt}{T(t,\alpha)} \, , \qquad V_R(t) = \frac{1}{2\pi} \int_{-R}^{R} E_\alpha(i\xi, t) \, \mathcal{F}_T^\alpha[f](\xi) \, d\xi,$$

then

$$\lim_{R\to\infty} \left\| U_R - \mathcal{F}_T^\alpha[f] \right\|_{L^2(\mathbb{R},\mu)} = 0 \quad \text{and} \quad \lim_{R\to\infty} \| V_R - f \|_{L^2(\mathbb{R},\mu)} = 0.$$

Proof. Let us prove item (2). Given $f \in L^2(\mathbb{R}, \mu)$, consider a sequence $\{f_n\} \subset L^1(\mathbb{R}, \mu) \cap L^2(\mathbb{R}, \mu)$ with

$$\lim_{n\to\infty} \| f - f_n \|_{L^2(\mathbb{R},\mu)} = 0, \qquad \lim_{n\to\infty} \left\| \mathcal{F}_T^\alpha[f] - \mathcal{F}_T^\alpha[f_n] \right\|_{L^2(\mathbb{R},\mu)} = 0.$$

Hence,

$$\lim_{n\to\infty} \| f_n \|_{L^2(\mathbb{R},\mu)} = \| f \|_{L^2(\mathbb{R},\mu)}, \qquad \lim_{n\to\infty} \left\| \mathcal{F}_T^\alpha[f_n] \right\|_{L^2(\mathbb{R})} = \left\| \mathcal{F}_T^\alpha[f] \right\|_{L^2(\mathbb{R})}.$$

Since Theorem 13 gives

$$\| f_n \|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \left\| \mathcal{F}_T^\alpha[f_n] \right\|_{L^2(\mathbb{R})}$$

for every n, we obtain Plancherel's theorem:

$$\| f \|_{L^2(\mathbb{R},\mu)} = \frac{1}{\sqrt{2\pi}} \left\| \mathcal{F}_T^\alpha[f] \right\|_{L^2(\mathbb{R})}.$$

Item (3) follows from item (2) by the polarization identity in Hilbert spaces.

The first and the last items can be easily proved. $\square$

Let us define now the generalized Fourier transform in $L^p(\mathbb{R},\mu)$ for any $1<p<2$. If $f\in L^p(\mathbb{R},\mu)$ for some $1<p<2$, then $f=f_1+f_2$ with

$$f_1 = f\chi_{_{\{|f|>1\}}}, \qquad f_2 = f\chi_{_{\{|f|\le 1\}}}.$$

Since

$$|f_1| \le |f|^p\chi_{_{\{|f|>1\}}} \le |f|^p \in L^1(\mathbb{R},\mu), \qquad |f_2|^2 \le |f|^p\chi_{_{\{|f|\le 1\}}} \le |f|^p \in L^1(\mathbb{R},\mu),$$

we have $f_1\in L^1(\mathbb{R},\mu)$ and $f_2\in L^2(\mathbb{R},\mu)$. Thus, $\mathcal{F}_T^\alpha[f_1]$ and $\mathcal{F}_T^\alpha[f_2]$ are defined, and it is natural to define

$$\mathcal{F}_T^\alpha[f] = \mathcal{F}_T^\alpha[f_1+f_2] = \mathcal{F}_T^\alpha[f_1] + \mathcal{F}_T^\alpha[f_2].$$

Let us prove now the following version of Hausdorff-Young inequality.

Theorem 15. Let $f:\mathbb{R}\to\mathbb{C}$ be a function in $f\in L^p(\mathbb{R},\mu)$ for some $1\le p\le 2$ and $0<\alpha\le 1$. If $2\le q\le\infty$ is the conjugate exponent of p (i.e., $1/p+1/q=1$), then $\mathcal{F}_T^\alpha[f]\in L^q(\mathbb{R})$ and

$$\left\|\mathcal{F}_T^\alpha[f]\right\|_{L^q(\mathbb{R})} \le (2\pi)^{1/q}\left\|f\right\|_{L^p(\mathbb{R},\mu)}.$$

Proof. Theorem 6 gives that $\mathcal{F}_T^\alpha : L^1(\mathbb{R},\mu)\to L^\infty(\mathbb{R})$ is a continuous linear map with norm 1:

$$\left\|\mathcal{F}_T^\alpha[f]\right\|_{L^\infty(\mathbb{R})} \le \left\|f\right\|_{L^1(\mathbb{R},\mu)}$$

for every $f\in L^1(\mathbb{R},\mu)$.

Theorem 14 gives that $\mathcal{F}_T^\alpha : L^2(\mathbb{R},\mu)\to L^2(\mathbb{R})$ is a continuous linear map with norm $(2\pi)^{1/2}$:

$$\left\|\mathcal{F}_T^\alpha[f]\right\|_{L^2(\mathbb{R})} \le (2\pi)^{1/2}\left\|f\right\|_{L^2(\mathbb{R},\mu)}$$

for every $f\in L^2(\mathbb{R},\mu)$.

Riesz-Thorin Interpolation Theorem gives that if

$$\begin{aligned}\frac{1}{p_\theta} &= \frac{1-\theta}{1} + \frac{\theta}{2} = 1-\frac{\theta}{2}, \\ \frac{1}{q_\theta} &= \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2},\end{aligned}$$

for $\theta\in[0,1]$, then $\mathcal{F}_T^\alpha : L^{p_\theta}(\mathbb{R},\mu)\to L^{q_\theta}(\mathbb{R})$ is a continuous linear map satisfying

$$\begin{aligned}\left\|\mathcal{F}_T^\alpha[f]\right\|_{L^{q_\theta}(\mathbb{R})} &\le 1^{1-\theta}\left((2\pi)^{1/2}\right)^\theta\left\|f\right\|_{L^{p_\theta}(\mathbb{R},\mu)} = (2\pi)^{\theta/2}\left\|f\right\|_{L^{p_\theta}(\mathbb{R},\mu)} \\ &= (2\pi)^{1/q_\theta}\left\|f\right\|_{L^{p_\theta}(\mathbb{R},\mu)}.\end{aligned}$$

Note that

$$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1-\frac{\theta}{2}+\frac{\theta}{2} = 1.$$

If we consider $p = p_\theta$, then $q = q_\theta$ and the result follows. $\square$

5 | FRACTIONAL HEAT EQUATION

We want to study the fractional heat equation with $0 < \alpha \leq 1$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial G_T^\alpha(\partial G_T^\alpha u)}{(\partial x^\alpha)^2}(x,t), \quad x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}. \end{cases}$$

By applying the generalized Fourier transform on the variable x, we obtain

$$\begin{cases} \mathcal{F}_T^\alpha\left[\dfrac{\partial u}{\partial t}\right](\xi,t) = \mathcal{F}_T^\alpha\left[\dfrac{\partial G_T^\alpha(\partial G_T^\alpha u)}{(\partial x^\alpha)^2}\right](\xi,t), \quad \xi \in \mathbb{R},\, t > 0, \\ \mathcal{F}_T^\alpha\,[u]\,(\xi,0) = \mathcal{F}_T^\alpha\,[f]\,(\xi), \quad \xi \in \mathbb{R}. \end{cases}$$

If we denote by $U(\xi,t) = \mathcal{F}_T^\alpha\,[u]\,(\xi,t)$, then we obtain, by using Theorem 3,

$$\begin{cases} \dfrac{\partial U}{\partial t}(\xi,t) = -\xi^2 U(\xi,t), \quad \xi \in \mathbb{R},\, t > 0, \\ U(\xi,0) = \mathcal{F}_T^\alpha\,[f]\,(\xi), \quad \xi \in \mathbb{R}. \end{cases}$$

Hence,

$$U(\xi,t) = A(\xi)\,e^{-\xi^2 t},$$

and since $U(\xi,0) = \mathcal{F}_T^\alpha\,[f]\,(\xi)$, we conclude

$$U(\xi,t) = \mathcal{F}_T^\alpha\,[f]\,(\xi)\,e^{-\xi^2 t}.$$

Proposition 2 gives

$$\mathcal{F}_T^\alpha\left[\frac{1}{\sqrt{4\pi a}}\,\exp\left(-\frac{1}{4a}\left(\int_0^x \frac{d\omega}{T(\omega,\alpha)}\right)^2\right)\right](\xi) = \exp\left(-a\xi^2\right).$$

Therefore, if

$$K_t(x) = \frac{1}{\sqrt{4\pi t}}\,\exp\left(-\frac{1}{4t}v(x)^2\right),$$

then Theorem 7 gives

$$\begin{aligned} U(\xi,t) &= \mathcal{F}_T^\alpha\left[K_t\right](\xi)\,\mathcal{F}_T^\alpha\,[f]\,(\xi), \\ u(x,t) = (K_t * f)(x) &= \int_{-\infty}^{\infty} K_t\Big(w\big(v(x) - v(\omega)\big)\Big)\,f(\omega)\,v'(\omega)\,d\omega \\ &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{4t}\big(v(x) - v(\omega)\big)^2\right)\,f(\omega)\,v'(\omega)\,d\omega \\ &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{4t}\big(v(x) - y\big)^2\right)\,f(w(y))\,dy. \end{aligned}$$

We have obtained the solution of the equation heuristically. Let us show that this is the solution of the problem, under weak hypotheses.

Theorem 16. Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R})$ and $0 < \alpha \leq 1$. Then the function

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{4t}\left(v(x) - y\right)^2\right) f(w(y))\, dy$$

satisfies

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial G_T^\alpha(\partial G_T^\alpha u)}{(\partial x^\alpha)^2}(x,t), \quad x \in \mathbb{R},\, t > 0.$$

Also,

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} \left|u(x,t) - f(x)\right|^p \frac{dx}{T(x,\alpha)} = 0.$$

and

$$\lim_{t \to 0^+} u(x,t) = f(x)$$

if f is continuous at x.

Proof. Since $f \in L^p(\mathbb{R})$, it is easy to see that differentiation under the integral sign is possible and so, we can check that u satisfies the fractional partial differential equation. Theorem 12 gives the other statements. □

6 | CONCLUSIONS

In this paper, we introduce the theory of a generalized Fourier transform and we state its main properties. In particular, we study:

(1) The corresponding convolution of functions. We show that it is defined when these functions satisfy weak integrability conditions, and we use it in order to construct appropriate approximations of the identity.

(2) The generalized Fourier transform of fractional derivatives.

(2) The inverse generalized Fourier transform.

(3) Plancherel formula, which allows to define this generalized Fourier transform in L^2.

(4) Hausdorff-Young inequality, which allows to define the generalized Fourier transform in L^p for any $1 < p < 2$.

(5) We provide a table of Fourier transforms.

(6) We show that this generalized Fourier transform is useful in the study of fractional partial differential equations, by solving the fractional heat equation on the real line.

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Author contributions

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The authors declare no potential conflict of interests.

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