Boundedness in a predator-prey system with prey-taxis and nonlinear gradient-dependent sensitivity

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Abstract

In this paper, we extend the gradient-dependent nonlinear sensitivity assumption of Keller-Segel-Navier-Stokes system [M. Winkler, Z. Angew. Math. Phys. 2021] to predator-prey and Keller-Segel systems in two dimensions. Under appropriate regularity assumption on the initial data, the global boundedness of classical solution is obtained.

BOUNDEDNESS IN A PREDATOR-PREY SYSTEM WITH PREY-TAXIS AND NONLINEAR GRADIENT-DEPENDENT SENSITIVITY

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ABSTRACT. In this paper, we extend the gradient-dependent nonlinear sensitivity assumption of Keller-Segel-Navier-Stokes system [M. Winkler, Z. Angew. Math. Phys. 2021] to predator-prey and Keller-Segel systems in two dimensions. Under appropriate regularity assumption on the initial data, the global boundedness of classical solution is obtained.

1. INTRODUCTION

In this paper, we consider the prey-taxis model with nonlinear sensitivity as following

$$\begin{cases} \frac{\partial u}{\partial t} = d_u \Delta u - \nabla \cdot (u\phi(|\nabla v|^2)\nabla v) + \gamma u g(v) - u f(u), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} = d_v \Delta v + h(v) - u g(v), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (u,v)(x,0) = (u_0,v_0)(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote the population density of predator and prey at position x and time t, respectively. The coefficients $d_u, d_v > 0$ are constants representing the diffusion rate of predator and prey respectively. The coefficient γ represents the conversion rate of the predator to the prey, which is also a positive constant. The term $-\nabla\cdot(u\phi(|\nabla v|^2)\nabla v)$ describes the prey-taxis (mobility) with coefficient $\phi(|\nabla v|^2)$. The function $ug(v)$ account for the inter-specific interaction, functions $uf(u)$ and $h(v)$ denote the intra-specific interactions. More specifically, $g(v)$ is often referred to as the functional response function, which represents the average capture rate of prey by predators. The nonlinear function ϕ is suitably smooth and meets

$$|\phi(s)| \le K_\phi \cdot (s+1)^{-\frac{\alpha}{2}} \text{ for all } s \ge 0 \tag{1.2}$$

with some $K_\phi > 0$ and $\alpha > 0$. Specifically, in the sequel, we always assume that $f(u), g(v), h(v)$ satisfy the following hypotheses:

- **(H1):** The function $f : [0,\infty) \to (0,\infty)$ is continuously differentiable and there exist two constants $\beta < 0$ and $\mu \ge 0$ such that $f(u) \ge \beta$ and $f'(u) \ge \mu$ for all $u \ge 0$, which implies that we have $f(u) \ge \beta + \mu u$, for all $u \ge 0$.
- **(H2):** The function $g(v) \in C^2([0,\infty)), g(0) = 0, g(v) > 0$ in $(0,\infty)$ and $g'(v) > 0$ on $[0,\infty)$.
- **(H3):** The function $h : [0,\infty) \to \mathbb{R}$ is continuously differentiable satisfying $h(0) = 0$, and there exist two constants $\eta, K > 0$ such that $h(v) \le \eta v$ for any $v \ge 0$, $h(K) = 0$ and $h(v) < 0$ for all $v > K$.
- **(H4):** The initial data $u_0 \in C^0(\overline{\Omega})$ is nonnegative, and that $v_0 \in W^{1,q}(\Omega)$ $(q > 2)$ is nonnegative with $v_0 \not\equiv 0$.

On the basis of above hypotheses, the main results of current works are stated as follows.

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Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is bounded domain with smooth boundary. Let $d_u, d_v, \gamma > 0$ be constants, $f(u), g(v), h(v)$ satisfy (H1)-(H3) and the initial data (u_0, v_0) satisfy (H4). Moreover, assume that $\phi \in C^2([0,\infty))$ satisfies* (1.2) *with some $K_\phi > 0$ and some $\alpha > 0$. Then the problem* (1.1) *has a unique global classical solution*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times [0,\infty)), \\ v \in C^0([0,\infty); W^{1,2q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times [0,\infty)) \end{cases}$$

with

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,2q}(\Omega)} \le C,$$

where $C > 0$ is a constant independent of time t, and in particular $0 \le v \le K_0$ with

$$K_0 := \max\left\{\|v_0\|_{L^\infty}, K\right\}. \tag{1.3}$$

Remark 1.2. *When $n = 1$, the global boundedness of solutions to* (1.1) *can be derived by a similar process as in the proof of Theorem 1.1.*

Remark 1.3. *Actually, considering the nonlinear sensitivity function $\phi(s) = \chi$ is a positive constant, the system* (1.1) *can be reduce to classical predator-prey system with prey-taxis. Specifically, Jin and Wang* [4] *showed that the solution is globally bounded if $\chi > 0$ in two dimensions. In the present work, Theorem 1.1 shows that the solution is globally bounded only if the sensitivity function $\phi(s)$ satisfies* (1.2).

Remark 1.4. *When $\phi(s) = \chi$ is a positive constant, Wu et al.* [10] *showed that the solution is globally bounded if χ is small in higher dimensions. Does the prey-taxis coefficient function $\phi(s)$ satisfies* (1.2) *to ensure the global boundedness of solutions when α is small enough in higher dimensions? This probelm will be tackled in a subsequent paper.*

Next, we consider the flux-limited Keller-Segel system with logistic source as following

$$\begin{cases} \frac{\partial u}{\partial t} = d_u \Delta u - \nabla \cdot (u\phi(|\nabla v|^2)\nabla v) + \omega u(1-\mu u), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} = d_v \Delta v - v + u, & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (u,v)(x,0) = (u_0,v_0)(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where u, v represent the density of the cell population and the concentration of the chemoattractant. We consider this problem in a bounded domain $\Omega \subset \mathbb{R}^2$ with zero-flux boundary condition.

In two dimensions, it is well known that any presence of a logistic source will be sufficient to suppress blow-up by ensuring that all solutions to (1.4) are global-in-time and uniformly bounded [5, 7, 11]. In the case of $\omega > 0$ and $n = 2$, we can obtain the existence of the global classical solutions of system (1.4) by appropriately modifying the proof of Theorem 1.1. Therefore, we omit this proof here. In the following, we describe the main conclusions of problem (1.4) by several remarks.

Remark 1.5. *When $n = 1$, the global boundedness of solutions to* (1.1) *and* (1.4) *can be derived by a similar process as in the proof of Theorem 1.1.*

Remark 1.6. *When $\omega = 0$, $n = 2$ and the nonlinear sensitivity function $\phi(|\nabla v|^2)$ satisfies* (1.2) *with $\alpha > 0$, the global boundedness of solutions to* (1.4) *can be derived through a process similar to that in ref.* [8].

The rest of paper is organized as follows. In Section 2, we mainly derive the local existence and uniqueness of solutions, as well as some useful tools. In Section 3, we obtain some priori estimates. The global existence and uniform boundedness of solutions are proved in Section 4.

2. Local existence and preliminaries

Firstly, the operator $-\Delta+1$ is sectorial in L^p and therefore possesses closed fractional powers $(-\Delta+1)^\vartheta, \vartheta \in (0,1)$, with dense domain $D((-\Delta+1)^\vartheta)$ (see [2] or [9]). If $m \in \{0,1\}, p \in [1,\infty]$ and $q \in (1,\infty)$, then there exist some constants $c>0$ such that

$$\|w\|_{W^{m,p}(\Omega)} \leq c\|(-\Delta+1)^\vartheta w\|_{L^q(\Omega)}, \tag{2.1}$$

for all $w \in D((-\Delta+1)^\vartheta)$, provided that $m - \frac{n}{p} < 2\vartheta - \frac{n}{q}$.

Moreover, for $p<\infty$ the associated heat semigroup $(e^{t\Delta})_{t\geq 0}$ map $L^p(\Omega)$ into $D((-\Delta+1)^\vartheta)$ in any of the spaces $L^q(\Omega)$ for $q \geq p$, and there exist $c>0$ and $\iota>0$ such that

$$\|(-\Delta+1)^\vartheta e^{t(\Delta-1)} w\|_{L^q(\Omega)} \leq ct^{-\vartheta-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-\iota t}\|w\|_{L^p(\Omega)}, \text{ for all } w \in L^p(\Omega).$$

Finally, for any $\varepsilon>0$ and given $p \in (1,\infty)$, there exists $C_\varepsilon>0$ such that

$$\|(-\Delta+1)^\vartheta e^{t\Delta}\nabla\cdot w\|_{L^p(\Omega)} \leq C_\varepsilon t^{-\vartheta-\frac{1}{2}-\varepsilon} e^{-\iota t}\|w\|_{L^p(\Omega)}, \text{ for all } w \in L^p(\Omega). \tag{2.2}$$

We first assert local-in-time existence of a classical solution.

Lemma 2.1 (Local existence)**.** *Let $d_u, d_v, \gamma > 0$ be constants, $f(u), g(v), h(v)$ satisfy (H1)-(H3) and the initial data (u_0, v_0) satisfy (H4). Suppose that $\phi \in C^2([0,\infty))$ satisfies* (1.2)*. Then there exist $T_{\max} \in (0,\infty]$ and unique function pair (u,v) satisfying that*

$$u \in C^0(\overline{\Omega}\times[0,T_{\max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))$$

and

$$v \in C^0([0,T_{\max}); W^{1,2q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))$$

with $u, v > 0$ for all $t>0$. Moreover,

$$\text{if } T_{\max}<\infty, \text{ then } \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,2q}(\Omega)}\right) \to \infty \text{ as } t \nearrow T_{\max}.$$

Before starting the proof process, we introduce a important lemma as following. The argument are quite standard, and so we omit the proof here.

Lemma 2.2. [4, Lemma 2.2] *Under the conditions in Theorem 1.1, the solution (u,v) of* (1.1) *satisfies*

$$u(x,t)>0, \ 0<v(x,t)\leq K_0, \ (x,t) \in \Omega\times(0,\infty),$$

where K_0 is defined by (1.3)*, and it further holds that*

$$\limsup_{t\to\infty} v(x,t) \leq K, \ x \in \overline{\Omega}.$$

Moreover, there is a constant B independent of t such that

$$\int_\Omega u \leq B, \ t>0.$$

Proof of the Lemma 2.1. i) **Existence**. Let $\Phi_1(u,v) = -uf(u) + u + \gamma ug(v)$ and $\Phi_2(u,v) = h(v) + v - ug(v)$. We consider the fixed point equation $(u,v) = \Psi(u,v)$, where

$$\begin{aligned}\Psi(u,v) \equiv& \begin{pmatrix} \Psi_1(u,v) \\ \Psi_2(u,v) \end{pmatrix}\\ :=& \begin{pmatrix} e^{-t(A_{d_u}+1)}u_0 - \int_0^t e^{-(t-s)(A_{d_u}+1)}\nabla\cdot(u\phi(|\nabla v|^2)\nabla v) + \int_0^t e^{-(t-s)(A_{d_u}+1)}\Phi_1(u,v) \\ e^{-t(A_{d_v}+1)}u_0 + \int_0^t e^{-(t-s)(A_{d_v}+1)}\Phi_2(u,v) \end{pmatrix}\end{aligned}$$

in the closed subset $Y := \{(u,v) \in X : \|(u,v)\|_X \le R\}$ of the space

$$X := C^0\left([0,T]; C^0(\bar{\Omega})\right) \times L^\infty\left((0,T); W^{1,p}(\Omega)\right)$$

with norm $\|(u,v)\|_X := \|u\|_{L^\infty(\Omega\times(0,T))} + \|v\|_{L^\infty((0,T);W^{1,p}(\Omega))}$. Furthermore, from assumptions (H1)-(H3), let $L_\phi(R), L_f(R), L_g(R)$ denote Lipschitz constants for ϕ, f and g on the interval $(-R,R)$, respectively.

Fixing $\vartheta_{01} \in (\frac{1}{r},\frac{1}{2})$ and $\varepsilon \in (0,\frac{1}{2}-\vartheta_{01})$, and using (2.1) and (2.2), we can obtain that

$$\begin{aligned}\|\Psi_1(u,v) - \Psi_1(\overline{u},\overline{v})\|_{L^\infty(\Omega)} \le& C\int_0^t \|(A_{d_u}+1)^{\vartheta_{01}} e^{-(t-s)(A_{d_u}+1)}\nabla\cdot\left(u\phi(|\nabla v|^2)\nabla v - \overline{u}\phi(|\nabla\overline{v}|^2)\nabla\overline{v}\right)\|_{L^r(\Omega)}\\ &+ C\int_0^t \|(A_{d_u}+1)^{\vartheta_{01}} e^{-(t-s)(A_{d_u}+1)}\left(\Phi_1(u,v) - \Phi_1(\overline{u},\overline{v})\right)\|_{L^r(\Omega)}\\ :=& I_1 + I_2. \end{aligned} \tag{2.3}$$

for all $(u,v), (\overline{u},\overline{v}) \in Y$.

Then we estimate the $L^\infty-$bound for each of I_1 and I_2 separately. For I_1, we have

$$\begin{aligned} I_1 \le& C\int_0^t (t-s)^{-\vartheta_{01}-\frac{1}{2}-\varepsilon}\|u\phi(|\nabla v|^2)\nabla v - \overline{u}\phi(|\nabla\overline{v}|^2)\nabla\overline{v}\|_{L^r(\Omega)}\\ \le& C\int_0^t (t-s)^{-\vartheta_{01}-\frac{1}{2}-\varepsilon}\|u-\overline{u}\|_{L^\infty(\Omega)}\|\phi(|\nabla v|^2)\|_{L^\infty(\Omega)}\|\nabla v\|_{L^r(\Omega)}\\ &+ C\int_0^t (t-s)^{-\vartheta_{01}-\frac{1}{2}-\varepsilon}\|\overline{u}\|_{L^\infty(\Omega)}\|\phi(|\nabla v|^2) - \phi(|\nabla\overline{v}|^2)\|_{L^\infty(\Omega)}\|\nabla v\|_{L^r(\Omega)}\\ &+ C\int_0^t (t-s)^{-\vartheta_{01}-\frac{1}{2}-\varepsilon}\|\overline{u}\|_{L^\infty(\Omega)}\|\phi(|\nabla\overline{v}|^2)\|_{L^\infty(\Omega)}\|\nabla v - \nabla\overline{v}\|_{L^r(\Omega)}\\ \le& C\left(2R\|\phi\|_{L^\infty(\Omega)} + 2RL_\phi(R)\right)T^{\frac{1}{2}-\vartheta_{01}-\varepsilon}\|(u,v) - (\overline{u},\overline{v})\|_X \end{aligned} \tag{2.4}$$

for all $t \in [0,T]$. For I_2, we have

$$\begin{aligned} I_2 \le& C\int_0^t (t-s)^{-\vartheta_{01}} e^{-\iota(t-s)}\left(\|f(\overline{u}) - f(u)\|_{L^r(\Omega)} + \|\overline{u}\left(g(v) - g(\overline{v})\right)\|_{L^r(\Omega)}\right)\\ &+ C\int_0^t (t-s)^{-\vartheta_{01}} e^{-\iota(t-s)}\left(\|u-\overline{u}\|_{L^r(\Omega)} + \|g(v)(u-\overline{u})\|_{L^r(\Omega)}\right)\\ \le& C\left(L_f + RL_g(R) + 1 + g(K_0)\right)T^{1-\vartheta_{01}}\|(u,v) - (\overline{u},\overline{v})\|_X \end{aligned} \tag{2.5}$$

for all $t \in [0,T]$. Therefore, taking (2.4) and (2.5) into (2.3), we have

$$\|\Psi_1(u,v) - \Psi_1(\overline{u},\overline{v})\|_{L^\infty(\Omega)} \le \mathscr{K}\|(u,v) - (\overline{u},\overline{v})\|_X, \tag{2.6}$$

where $\mathscr{K} = C\left(2R\|\phi\|_{L^\infty(\Omega)} + 2RL_\phi(R)\right)T^{\frac{1}{2}-\vartheta_{01}-\varepsilon} + C\left(L_f + RL_g(R) + 1 + g(K_0)\right)T^{1-\vartheta_{01}}$.

Set $m=1, p=q=r>2$ in (2.1). Fixing $\vartheta_{02}\in(\frac{1}{2},1)$, we have

$$\begin{aligned}\|\Psi_2(u,v)-\Psi_2(\overline{u},\overline{v})\|_{W^{1,r}(\Omega)} \leq& C\int_0^t \|(A_{d_v}+1)^{\vartheta_{02}}e^{(t-s)(A_{d_v}+1)}(\Psi_2(u,v)-\Psi_2(\overline{u},\overline{v}))\|_{L^r(\Omega)}\\ \leq& C\int_0^t (t-s)^{-\vartheta_{02}}e^{-\iota(t-s)}\left(\|v-\overline{v}\|_{L^r(\Omega)}+\|h(v)-h(\overline{v})\|_{L^r(\Omega)}\right)\\ &+C\int_0^t (t-s)^{-\vartheta_{02}}e^{-\iota(t-s)}\left(\|\overline{u}(g(v)-g(\overline{v}))\|_{L^r(\Omega)}+\|g(v)(u-\overline{u})\|_{L^r(\Omega)}\right)\\ \leq& C\left(1+L_h(R)+RL_g(R)+g(K_0)\right)T^{1-\vartheta_{02}}\|(u,v)-(\overline{u},\overline{v})\|_X \quad (2.7)\end{aligned}$$

for all $t\in[0,T]$. Inserting $(\overline{u},\overline{v})\equiv(0,0)$ into (2.6) and (2.7), in particular, we obtain that Ψ maps X into itself if we choose R sufficiently large and then T small; apart from that, (2.6) and (2.7) show that after possibly diminishing T, Ψ becomes a contraction. Hence, we obtain the existence of $(u,v)\in X$ satisfying $(u,v)=\Psi(u,v)$. From standard parabolic regularity arguments [6] it follows that (u,v) satisfies the regularity properties listed in the formulation of the lemma, and solves (1.1) classically.

ii) **Uniqueness**. Assume that the system (1.1) has two different solutions (u,v) and $(\overline{u},\overline{v})$ in $\Omega\times[0,T]$. Let $U=u-\overline{u}, V=v-\overline{v}$ for $t\in(0,T)$ and using Lemma 2.2 and the Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega U^2+d_u\int_\Omega|\nabla U|^2=&\int_\Omega U\phi(|\nabla v|^2)\nabla U\cdot\nabla v+\int_\Omega\overline{u}(\phi(|\nabla v|^2)-\phi(|\nabla\overline{v}|^2))\nabla U\cdot\nabla v\\ &+\int_\Omega\overline{u}\phi(|\nabla\overline{v}|^2)\nabla U\cdot\nabla V-\int_\Omega\left((f(u)-f(\overline{u}))\overline{u}+f(u)U\right)U\\ &+\gamma\int_\Omega\left(g(v)U+\overline{u}(g(v)-g(\overline{v}))\right)U\\ \leq& K_\phi R(R^2+1)^{-\frac{\alpha}{2}}\int_\Omega U\nabla U+K_\phi R(R^2+1)^{-\frac{\alpha}{2}}\int_\Omega\nabla U\cdot\nabla V\\ &+2R^3\int_\Omega\nabla U\cdot\nabla V+\gamma g(K_0)\int_\Omega U^2+\gamma g'(K_0)R\int_\Omega UV\\ &+L_fR\int_\Omega U^2-\beta\int_\Omega U^2\\ \leq&\left(\frac{3K_\phi^2R^2(R^2+1)^{-\alpha}}{4d_u}+\gamma g(K_0)+L_fR\right)\int_\Omega U^2+d_u\int_\Omega|\nabla U|^2\\ &+\frac{3}{4d_u}\left(K_\phi^2R^2(R^2+1)^{-\alpha}+R^6\right)\int_\Omega|\nabla V|^2+\frac{(\gamma g'(K_0)R)^2}{4\beta}\int_\Omega V^2.\end{aligned} \tag{2.8}$$

And for all $t\in(0,T)$, we have

$$\begin{aligned}-\int_\Omega\left((f(u)-f(\overline{u}))\overline{u}+f(u)U\right)U\leq&\int_\Omega\overline{u}|f(u)-f(\overline{u})|U-\int_\Omega f(u)U^2\\ \leq& L_fR\int_\Omega U^2-\beta\int_\Omega U^2,\end{aligned}$$

because of assumptions (H1). Similarly, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla V|^2 + d_v\int_\Omega |\Delta V|^2 \leq &\int_\Omega |h(\overline{v}) - h(v)|\Delta V + \int_\Omega \overline{u}(g(\overline{v}) - g(v))\Delta V + \int_\Omega (\overline{u} - u)g(v)\Delta V \\ \leq &(L_h + Rg'(K_0))\int_\Omega \nabla V\Delta V + g(K_0)\int_\Omega U\Delta V \\ \leq &\frac{(L_h + Rg'(K_0))^2}{2d_v}\int_\Omega |\nabla V|^2 + d_v\int_\Omega |\Delta V|^2 + \frac{g^2(K_0)}{2d_v}\int_\Omega U^2. \quad (2.9)\end{aligned}$$

Finally, adding to (2.8) and (2.9) yields

$$\frac{d}{dt}\left(\int_\Omega U^2 + \int_\Omega |\nabla V|^2\right) \leq \beta'\left(\int_\Omega U^2 + \int_\Omega |\nabla V|^2\right) + C, \text{ for all } t \in (0, T),$$

where $\beta' = \max\left\{\frac{3K_\phi^2 R^2(R^2+1)^{-\alpha}}{4d_u} + \gamma g(K_0) + L_f R + \frac{g^2(K_0)}{2d_v}, \frac{3}{4d_u}\left(K_\phi^2 R^2(R^2+1)^{-\alpha} + R^6\right) + \frac{(L_h+Rg'(K_0))^2}{2d_v}\right\}$.

It follows from Gronwall's lemma that $U \equiv 0, V \equiv 0$ in $\Omega \times (0, T)$ and hence $(u, v) \equiv (\overline{u}, \overline{v})$ in $\Omega \times (0, T)$. $\square$

Lemma 2.3. [8, Corollary 4.2] *Assume that $q > 1$ and $\xi > 0$ satisfying $\xi < 2q + 2$. Then for all $\delta > 0$ one can find $C(\delta) := C(\delta, q, \xi) > 0$ such that for all $\varphi \in C^2(\overline{\Omega})$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$,*

$$\int_\Omega |\nabla\varphi|^\xi \leq \delta\int_\Omega |\nabla\varphi|^{2q-2}|D^2\varphi|^2 + C(\delta).$$

Lemma 2.4 (Gagliardo-Nirenberg inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $1 \leq p,\ q \leq \infty$ satisfying $(n - kq)p < nq$ for some $k > 0$ and $r \in (0, p)$. Then, for any $\varphi \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two constants C_{GN} and c_1 depending only on Ω, q, k, r and N such that*

$$\|\varphi\|_{L^p} \leq C_{GN}\|D^k\varphi\|_{L^q}^a\|\varphi\|_{L^r}^{1-a} + c_1\|\varphi\|_{L^r},$$

where $a \in (0, 1)$ fulfilling

$$\frac{1}{p} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1 - a)\frac{1}{r}.$$

3. Prior estimate of solutions

In this section, we are devoted to proving Theorem 1.1 by deriving some a priori estimate.

3.1. Case I: $\alpha \in (0, 1)$.

Lemma 3.1. *Suppose* (1.2) *holds with some $K_\phi > 0$ and $\alpha \in (0, 1)$. Let $p > 2, q > 1$ and $q + \alpha > 2$. Then there exists $C_1 > 0$ such that*

$$\frac{d}{dt}\int_\Omega u^p + p\beta\int_\Omega u^p + p\mu\int_\Omega u^{p+1} + \frac{d_u p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \leq p\zeta_1\int_\Omega u^{p\theta} + p\zeta_2\int_\Omega |\nabla v|^{\frac{2(1-\alpha)\theta}{\theta-1}} + pC_1,$$

for all $t \in (0, T_{\max})$, where

$$\theta = \frac{p(q-1) - 2(1-\alpha)}{p(q+\alpha-2)}, C_1 = \frac{\theta\gamma g(K_0)}{\theta - 1}|\Omega|$$

and

$$\zeta_1 = \gamma\frac{g(K_0)}{\theta} + \frac{K_\phi^2(p-1)}{2d_u}, \zeta_2 = \frac{K_\phi^2(p-1)}{2d_u}.$$

Proof. Using the positivity of of u in $\overline{\Omega}\times(0,T_{\max})$, we multiply both ends of the first equation of (1.1) by u^{p-1} at the same time and integrate it on Ω. And then using Young's inequality along with (1.2), we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p =& -d_u(p-1)\int_\Omega u^{p-2}|\nabla u|^2+(p-1)\int_\Omega u^{p-1}\phi(|\nabla v|^2)\nabla u\cdot\nabla v\\ &-\int_\Omega u^p f(u)+\gamma\int_\Omega u^p g(v)\\ \le&-\frac{d_u(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2-\int_\Omega u^p(\beta+\mu u)+\gamma g(K_0)\int_\Omega u^p\\ &+\frac{K_\phi^2(p-1)}{2d_u}\int_\Omega u^p(|\nabla v|^2+1)^{-\alpha}|\nabla v|^2. \end{aligned}\tag{3.1}$$

Let $\theta:=\frac{p(q-1)-2(1-\alpha)}{p(q+\alpha-2)}$ and θ satisfies

$$\theta-1=\frac{(p-2)(1-\alpha)}{p(q+\alpha-2)}>0.$$

Using the Young's inequality, we have

$$\int_\Omega u^p(|\nabla v|^2+1)^{-\alpha}|\nabla v|^2\le\int_\Omega u^p|\nabla v|^{2(1-\alpha)}\le\int_\Omega u^{p\theta}+\int_\Omega|\nabla v|^{\frac{2(1-\alpha)\theta}{\theta-1}}\tag{3.2}$$

and

$$\int_\Omega u^p\le\frac{1}{\theta}\int_\Omega u^{p\theta}+\frac{\theta}{\theta-1}|\Omega|.\tag{3.3}$$

Substitute (3.2) and (3.3) into (3.1). Then the proof is completed. $\square$

Lemma 3.2. *Assume that $p>2$, $q>1$, $\alpha\in(0,1)$ and $p+\alpha>2$. Then there exists $C>0$ such that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega|\nabla v|^{2q}+qd_v\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2\le&2q\|f'\|_{C([0,K_0])}\int_\Omega|\nabla v|^{2q}+2q\eta_1\int_\Omega u^{2\lambda}\\ &+2q\eta_2\int_\Omega|\nabla v|^{\frac{2(q-1)\lambda}{\lambda-1}}+C\end{aligned}$$

holds for all $t\in(0,T_{\max})$, where

$$\mathcal{K}:=\|h'\|_{C([0,K_0])},\ \eta_1=\frac{qg(K_0)}{d_v\lambda}\ \text{and}\ \eta_2=\frac{q(\lambda-1)g(K_0)}{d_v\lambda}.$$

Proof. By a direct calculation, we see that

$$\begin{aligned}\frac{1}{2q}\frac{d}{dt}\int_\Omega|\nabla v|^{2q}=&\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla(d_v\Delta v+h(v)-ug(v))\\ =&d_v\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla\Delta v+\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla h(v)\\ &-\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla(ug(v))\\ :=&I_1+I_2+I_3.\end{aligned}\tag{3.4}$$

Specifically, recalling the fact about $\nabla v\cdot\nabla(\Delta v)=\frac{1}{2}\Delta|\nabla v|^2-|D^2v|^2$ and Green's formulas $\int_\Omega \nabla u\cdot\nabla v=-\int_\Omega u\Delta v+\int_{\partial\Omega}\frac{\partial v}{\partial\nu}udS$, we can see that

$$
\begin{aligned}
I_1 =&d_v\int_\Omega |\nabla v|^{2(q-1)}\nabla v\cdot\nabla\Delta v\\
=&\frac{d_v}{2}\int_\Omega |\nabla v|^{2(q-1)}\Delta|\nabla v|^2-d_v\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2\\
=&\frac{d_v}{2}\int_{\partial\Omega}|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}-\frac{d_v(q-1)}{2}\int_\Omega|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2\\
&-d_v\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2.
\end{aligned}\tag{3.5}
$$

Next we deal with the integration on the boundary $\partial\Omega$. Using an argument in [3, Lemma 3.2], we can find $C_2>0$ such that

$$
\begin{aligned}
\frac{1}{2}\int_{\partial\Omega}|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\leq&\frac{q-1}{q^2}\int_\Omega|\nabla|\nabla v|^q|^2+C_2\\
=&\frac{q-1}{4}\int_\Omega|\nabla v|^{2(q-2)}\left|\nabla|\nabla v|^2\right|^2+C_2
\end{aligned}\tag{3.6}
$$

holds for all $t\in(0,T_{\max})$. Applying (3.6) to (3.5), we obtain

$$
I_1\leq d_vC_2-\frac{d_v(q-1)}{4}\int_\Omega|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2-d_v\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2.\tag{3.7}
$$

It follows from the hypothesis (A2) that

$$
I_2=\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla h(v)=\int_\Omega h'(v)|\nabla v|^{2q}\leq\mathcal{K}\int_\Omega|\nabla v|^{2q}.\tag{3.8}
$$

In view of Young's inequality and recalling $|\Delta v|^2\leq 2|D^2v|^2$, we have

$$
\begin{aligned}
I_3=&-\int_\Omega|\nabla v|^{2(q-1)}\nabla v\cdot\nabla(ug(v))\\
=&(q-1)\int_\Omega ug(v)|\nabla v|^{2(q-2)}\nabla|\nabla v|^2\cdot\nabla v+\int_\Omega ug(v)|\nabla v|^{2(q-1)}\Delta v\\
\leq&\frac{d_v(q-1)}{4}\int_\Omega|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2+\frac{(q-1)}{d_v}\int_\Omega u^2g^2(v)|\nabla v|^{2(q-1)}\\
&+\frac{d_v}{2}\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2+\frac{1}{d_v}\int_\Omega u^2g^2(v)|\nabla v|^{2(q-1)}\\
\leq&\frac{d_v(q-1)}{4}\int_\Omega|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2+\frac{d_v}{2}\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2\\
&+\frac{q}{d_v}g(K_0)\int_\Omega u^2|\nabla v|^{2(q-1)}\\
\leq&\frac{d_v(q-1)}{4}\int_\Omega|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2+\frac{d_v}{2}\int_\Omega|\nabla v|^{2(q-1)}|D^2v|^2\\
&+\frac{qg(K_0)}{d_v\lambda}\int_\Omega u^{2\lambda}+\frac{q(\lambda-1)g(K_0)}{d_v\lambda}\int_\Omega|\nabla v|^{\frac{2(q-1)\lambda}{\lambda-1}},
\end{aligned}\tag{3.9}
$$

where

$$\lambda = \frac{p(q-1)-2(1-\alpha)}{2(q+\alpha-2)} \text{ and } \frac{(q-1)\lambda}{\lambda-1} = \frac{p(q-1)-2(1-\alpha)}{(p-2)}.$$

Here we note that our hypothesis $p > 2$ and $q > 1$ ensure that λ satisfies

$$\lambda - 1 = \frac{(p-2)(q-1)}{2(q+\alpha-2)} > 0.$$

Plugging (3.7)-(3.9) into (3.4), one has that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + qd_v \int_\Omega |\nabla v|^{2(q-1)}|D^2 v|^2 \leq & 2q\mathcal{K}\int_\Omega |\nabla v|^{2q} + \frac{2q^2 g(K_0)}{d_v\lambda}\int_\Omega u^{2\lambda} \\ & + \frac{2q^2(\lambda-1)g(K_0)}{d_v\lambda}\int_\Omega |\nabla v|^{\frac{2(q-1)\lambda}{\lambda-1}} + 2qd_v C_2, \end{aligned} \quad (3.10)$$

for all $t \in (0, T_{\max})$. This proof is completed. □

Lemma 3.3. *Let $\alpha \in (0,1), p > 2$ and $q > 2-\alpha$ along with*

$$p(1-\alpha) < q - \alpha. \quad (3.11)$$

Then for all $\delta_1 > 0$ there exists $C := C(\delta_1) > 0$ such that

$$\int_\Omega u^{p\theta} \leq \delta_1 \int_\Omega u^{p-2}|\nabla u|^2 + C(\delta_1) \quad (3.12)$$

holds, for all $t \in (0, T_{\max})$.

Proof. From (3.5), we know that $\theta > 1$. Applying the Gagliardo-Nirenberg inequality and Lemma 2.2, we have

$$\begin{aligned}\int_\Omega u^{p\theta} =& \|u^{\frac{p}{2}}\|^{2\theta}_{L^{2\theta}(\Omega)} \\ \leq & C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|^{2\theta\beta_1}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2\theta(1-\beta_1)}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^{2\theta}_{L^{\frac{2}{p}}(\Omega)}\right) \\ =& C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|^{2\theta\beta_1}_{L^2(\Omega)}\|u\|^{p\theta(1-\beta_1)}_{L^1(\Omega)} + \|u\|^{p\theta}_{L^1(\Omega)}\right) \\ \leq & C_{GN}B^{p\theta(1-\beta_1)}\|\nabla u^{\frac{p}{2}}\|^{2\theta\beta_1}_{L^2(\Omega)} + C_{GN}B^{p\theta}, \end{aligned} \quad (3.13)$$

where $\beta_1 = \frac{p\theta-1}{p\theta} \in (0,1)$. Furthermore, due to (3.11), one has that

$$2\theta\beta - 2 = \frac{2}{p} \times \frac{p(1-\alpha)-q+\alpha}{q+\alpha-2} < 0. \quad (3.14)$$

Therefore, we can obtain (3.12) as consequences of (3.13) and (3.14). □

Lemma 3.4. *Assume (1.2) holds with some $K_\phi > 0$ and $\alpha \in (0,1)$. Let $p > 2$ and $q > 2-\alpha$ along with*

$$q < p - \alpha.$$

Then for any given $\delta_2 > 0$, one can pick $C := C(\delta_2) > 0$ such that

$$\int_\Omega |\nabla v|^{\frac{(2q-2)\lambda}{\lambda-1}} \leq \delta_2 \int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 + C(\delta_2)$$

holds, for any $t \in (0, T_{\max})$.

Proof. We only need to verify $\frac{(2q-2)\lambda}{\lambda-1} > 0$ and $\lambda < 2q+2$ by Lemma 2.3, according to the assumptions $q > 2-\alpha$ and $q < p-\alpha$, we have

$$p(q-1) - 2(1-\alpha) > (p-2)(1-\alpha) > 0$$

and

$$2\frac{p(q-1)-2(1-\alpha)}{p-2} - 2(q+1) = \frac{4(q+\alpha-p)}{p-2} < 0.$$

This proof is complete. □

Lemma 3.5. *Assume that $p > 2$, $q > 1$ and $\alpha \in (0,1)$ satisfying*

$$q+\alpha < p < \min\left\{\frac{q-\alpha}{1-\alpha}, 2(q+\alpha-1)\right\}, \tag{3.15}$$

then

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \le C,$$

for all $t \in (0, T_{\max})$.

Proof. Combining lemma 3.1 with lemma 3.2, one has that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + \frac{d_u p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \\
&\quad + q d_v \int_\Omega |\nabla v|^{2(q-1)}|D^2 v|^2 + p\beta\int_\Omega u^p + p\mu\int_\Omega u^{p+1} \\
&\le (p\zeta_1+\eta_1)\int_\Omega u^{p\theta} + (p\zeta_2+\eta_2)\int_\Omega |\nabla v|^{\frac{2(1-\alpha)\theta}{\theta-1}} \\
&\quad + 2q\mathcal{K}\int_\Omega |\nabla v|^{2q} + C
\end{aligned} \tag{3.16}$$

for all $t \in (0, T_{\max})$. We can find $\delta_1 = \frac{d_u p(p-1)}{2(p\zeta_1+\eta_1)}$ by Lemma 3.3 such that δ_1 fulfilling

$$\begin{aligned}
\int_\Omega u^{p\theta} \le& \delta_1\int_\Omega u^{p-2}|\nabla u|^2 + C(\delta_1) \\
\le& \delta_1\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p\mu}{p\zeta_1+\eta_1}\int_\Omega u^{p+1} + C(\delta_1)
\end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max})$. Furthermore, by the fact (3.15), we have

$$\frac{p(q-1)-2(1-\alpha)}{(p-2)} - q = \frac{2(q+\alpha-1)-p}{p-2} > 0.$$

Using Lemma 3.4 we can find $\delta_2 := \frac{q d_v}{p\zeta_2+\eta_2}$ fulfilling

$$\int_\Omega |\nabla v|^{\frac{2(1-\alpha)\theta}{\theta-1}} \le \delta_2\int_\Omega |\nabla v|^{2(q-1)}|D^2 v|^2 + C(\delta_2), \tag{3.18}$$

for all $t \in (0, T_{\max})$. Substituting (3.17) and (3.18) into (3.16), we have

$$\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + \gamma_1\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) \le C(\delta_1) + C(\delta_2) + C,$$

where $\gamma_1 := \min\{p\beta, \mathcal{K}\}$, which entails $\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \le C$ with $C > 0$. This completes the proof of the lemma. □

Remark 3.6. (3.11) *is also equivalent to* $p\theta - (p+1) = \frac{p(1-\alpha)-q+\alpha}{q+\alpha-2} < 0$.

3.2. **Case II:** $\alpha \geq 1$.

Lemma 3.7. *Assume* (1.2) *holds with some* $K_\phi > 0$ *and* $\alpha > 1$*, and let* $p > 2$ *and* $q > 1$*. Then there exists* $\eta_3 > 0$ *such that*

$$\frac{d}{dt}\int_\Omega u^p + \frac{d_u p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + p\beta\int_\Omega u^p + p\mu\int_\Omega u^{p+1} \leq p\zeta_3\int_\Omega u^p, \tag{3.19}$$

holds for all $t \in (0, T_{\max})$*, where* $\zeta_3 := \gamma g(K_0) + \frac{K_\phi^2(p-1)}{2d_u}$.

Proof. We make use of the positively of u in $\overline{\Omega} \times (0, T_{\max})$, integrating the first equation of (1.1) on Ω and using Young's inequality along with (1.2). Then we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p =& - d_u(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\int_\Omega u^{p-1}\phi(|\nabla v|^2)\nabla u \cdot \nabla v \\ &- \int_\Omega u^p f(u) + \gamma\int_\Omega u^p g(v) \\ \leq& - \frac{d_u(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \gamma g(K_0)\int_\Omega u^p \\ &+ \frac{K_\phi^2(p-1)}{2d_u}\int_\Omega u^p(|\nabla v|^2+1)^{-\alpha}|\nabla v|^2,\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{\max})$. Here if $\alpha > 1$, we have

$$\frac{|\nabla v|^2}{(|\nabla v|^2+1)^\alpha} < \frac{|\nabla v|^2}{|\nabla v|^2+1} < 1.$$

Then we have

$$\frac{K_\phi^2}{2d_u(p-1)}\int_\Omega u^p(|\nabla v|^2+1)^{-\alpha}|\nabla v|^2 \leq \frac{K_\phi^2}{2d_u(p-1)}\int_\Omega u^p. \tag{3.21}$$

Next, substitute (3.21) into (3.20), which implies that (3.19) holds. □

Lemma 3.8. *Let* $p > 2$*. Then for any* $\delta_3 > 0$ *there exists* $C(\delta_3) > 0$ *such that*

$$\int_\Omega u^p \leq \delta_3\int_\Omega |u|^{p-2}|\nabla u|^2 + C(\delta_3)$$

holds for all $t \in (0, T_{\max})$.

Proof. Applying Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_\Omega u^p =& \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\ \leq& C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right) \\ \leq& C_{GN}B^{\frac{2}{p}}\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)} + C_{GN}B^2.\end{aligned} \tag{3.22}$$

We can get the conclusion immediately from (3.22) and $\frac{2(p-1)}{p} - 2 < 0$. □

Lemma 3.9. *Assume that $p>2$ and $q>1$. Then there exists $C>0$ such that*

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + qd_v\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 \le 2q\mathcal{K}\int_\Omega |\nabla v|^{2q} + 2q\eta_3\int_\Omega u^p + 2q\eta_4\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} + C \tag{3.23}$$

holds for all $t\in(0,T_{\max})$, where

$$\eta_3 = \frac{2qg(K_0)}{d_vp} \text{ and } \eta_4 = \frac{q(p-2)g(K_0)}{d_vp}.$$

Proof. Let $\lambda=\frac{p}{2}$ in (3.10). We can obtain (3.23) immediately. □

Lemma 3.10. *Assume (1.2) holds with some $K_\phi>0$. Let $p>2$, $q>1$ and $p>q+1$. Then for any given $\delta_4>0$, there exists $C:=C(\delta_4)>0$ such that*

$$\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} \le \delta_4\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 + C(\delta_4)$$

holds.

Proof. It is sufficient to verify $\frac{2p(q-1)}{p-2}>0$ and $\frac{2p(q-1)}{p-2}-2(q+1)=\frac{2(1+q-p)}{p-2}<0$ by Lemma 2.3. In fact, that is obvious by assumptions $q>2$ and $p>q+1$. □

Lemma 3.11. *Assume that $p>2$, $q>1$ and $\alpha\ge 1$ with $p\in(q+1,2q)$. Then there exists $C>0$ such that*

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} < C.$$

holds for all $t\in(0,T_{\max})$.

Proof. Collecting the estimates of (3.19) and (3.23), we end up with

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + \frac{d_up(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\\
&\quad + qd_v\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 + p\beta\int_\Omega u^p + p\mu\int_\Omega u^{p+1}\\
\le&(p\zeta_3+2q\eta_3)\int_\Omega u^p + 2q\mathcal{K}\int_\Omega |\nabla v|^{2q} + 2q\eta_4\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} + C.
\end{aligned}\tag{3.24}$$

Then (3.24) implies that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + p\beta\left(\int_\Omega |\nabla v|^{2q} + \int_\Omega u^p\right)\\
&\quad + \frac{d_up(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + qd_v\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 + p\mu\int_\Omega u^{p+1}\\
\le&(p\zeta_3+2q\eta_3)\int_\Omega u^p + (2q\mathcal{K}+p\beta)\int_\Omega |\nabla v|^{2q} + 2q\eta_4\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} + C
\end{aligned}\tag{3.25}$$

for all $t\in(0,T_{\max})$. Since $\frac{p(q-1)}{p-2}-q=\frac{2q-p}{p-2}>0$, then by Lemma 3.8, there exists a constant $C_1>0$ such that

$$\int_\Omega u^p \le \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^{p+1} + C_1 \tag{3.26}$$

and

$$(2q\mathcal{K}+q\beta)\int_\Omega |\nabla v|^{2q} + 2q\eta_4\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} \leq (2q\mathcal{K}+2q\eta_4+1)\int_\Omega |\nabla v|^{\frac{2p(q-1)}{p-2}} \leq \int_\Omega |\nabla v|^{2(q-1)}|D^2 v|^2 + C_1, \tag{3.27}$$

for all $t\in(0,T_{\max})$.

Plugging (3.26) and (3.27) into (3.25), we conclude that

$$\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + p\beta\left(\int_\Omega |\nabla v|^{2q} + \int_\Omega u^p\right) \leq C_2,$$

which entails $\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \leq C_2$, where C_2 is a constant with $C_2>0$. This completes the proof of the lemma. ☐

4. Proof of Theorem 1.1

Lemma 4.1. *If $K_\phi>0$ and $\alpha>0$ in (1.2), then there exists a constant $C>0$ fulfilling*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \text{ for all } t\in(0,T_{\max}).$$

Proof. We use semigroup arguments to obtain the L^∞-bound of u. Let $\tau\in(0,T_{\max})$ be given such that $\tau<1$. Next, using the the variation of constants formula, we have

$$\begin{aligned} u(\cdot,t) =& e^{-t(A_{d_u}+1)}u_0 - \int_0^t e^{-(t-s)(A_{d_u}+1)}\nabla\cdot(u\phi(|\nabla v|^2)\nabla v) + \int_0^t e^{-(t-s)(A_{d_u}+1)}\Phi_1(u,v) \\ :=& J_1+J_2+J_3, \end{aligned}$$

where $A_{d_u}=-d_u\Delta$ and $\Phi_1(u(\cdot,t),v(\cdot,t)) = -uf(u)+u+\gamma u g(v)$. Then we estimate the L^∞-bound for each of J_1, J_2 and J_3 separately.

For J_1, set $m=0, p=q=\infty$ in (2.1), we find that

$$\|J_1\|_{L^\infty(\Omega)} \leq \|(A_{d_u}+1)^{\vartheta_1}e^{-t(A_{d_u}+1)}u_0\|_{L^\infty(\Omega)} \leq ct^{-\vartheta_1}e^{-\iota t}\|u_0\|_{L^\infty(\Omega)}, \text{ for all } t\in(\tau,T_{\max}), \tag{4.1}$$

where $\vartheta_1\in(0,1)$ and $\iota>0$.

For J_2, set $m=0$ in (2.1) and $r>2$. So we can choose $\vartheta_2\in(\frac{1}{r},\frac{1}{2})$ and $\varepsilon\in(0,\frac{1}{2}-\vartheta_2)$, and hence we have

$$\begin{aligned} \|J_2\|_{L^\infty(\Omega)} \leq& C\|(A_{d_u}+1)^{\vartheta_2}J_2\|_{L^r(\Omega)} \\ \leq& C\int_0^t \|(A_{d_u}+1)^{\vartheta_2}e^{-(t-s)(A_{d_u}+1)}\nabla\cdot(u\phi(|\nabla v|^2)\nabla v)\|_{L^r(\Omega)} \\ \leq& C\int_0^t (t-s)^{-\vartheta_2-\frac{1}{2}-\varepsilon}e^{-\iota(t-s)}\|u\phi(|\nabla v|^2)\nabla v)\|_{L^r(\Omega)}, \text{ for all } t\in(0,T_{\max}). \end{aligned}$$

Case 1: If $\alpha\in(0,1)$ and $r>\frac{q+p(1-\alpha)}{1-\alpha}$, using the Hölder inequality and Lemma 3.5, we have

$$\begin{aligned} \|u\phi(|\nabla v|^2)\nabla v)\|_{L^r(\Omega)} \leq& K_\phi\cdot\|u|\nabla v|^{2(1-\alpha)}\|_{L^r(\Omega)} \\ \leq& K_\phi\cdot\left(\int_\Omega 1^{\frac{(r-p)(1-\alpha)-q}{r(1-\alpha)}}\right)^{\frac{(1-\alpha)}{(r-p)(1-\alpha)-q}}\left(\int_\Omega u^p\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla v|^{2q}\right)^{\frac{1-\alpha}{q}} \\ \leq& C, \text{ for all } t\in(\tau,T_{\max}). \end{aligned} \tag{4.2}$$

Case 2: If $\alpha \geq 1$ and $r > p + 2q$, using the Hölder inequality and Lemma 3.11, we have

$$\begin{aligned} \|u\phi(|\nabla v|^2)\nabla v)\|_{L^r(\Omega)} \leq & K_\phi \cdot \|u\nabla v\|_{L^r(\Omega)} \\ \leq & K_\phi \cdot \left(\int_\Omega 1^{\frac{r-p-2q}{r}}\right)^{\frac{1}{r-p-2q}} \left(\int_\Omega u^p\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla v|^{2q}\right)^{\frac{1}{2q}} \\ \leq & C, \text{ for all } t \in (\tau, T_{\max}). \end{aligned} \tag{4.3}$$

Therefore, From (4.2)(case $\alpha \in (0,1)$) or (4.3)(case $\alpha \geq 1$), we obtain that for all $t \in (\tau, T_{\max})$,

$$\begin{aligned} \|J_2\|_{L^\infty(\Omega)} \leq & C \int_0^t (t-s)^{-\vartheta_2 - \frac{1}{2} - \varepsilon} e^{-\iota(t-s)} \\ \leq & C \int_0^\infty \sigma^{-\vartheta_2 - \frac{1}{2} - \varepsilon} e^{-\iota\sigma} \\ \leq & C\Gamma\left(\frac{1}{2} - \vartheta_2 - \varepsilon\right), \end{aligned} \tag{4.4}$$

where $\Gamma(x)$ is the Gamma function and $\iota > 0$. Since $\frac{1}{2} - \vartheta_2 - \varepsilon > 0$, the $\Gamma\left(\frac{1}{2} - \vartheta_2 - \varepsilon\right)$ is positive and real.

For J_3, set $m = 1, l \in (2, \infty)$ and $p > 2$, so we can choose $\vartheta_3 \in \left(\frac{1}{2}(1 - \frac{2}{l} - \frac{2}{p}), 1\right)$. By using Lemma 3.5, we have

$$\begin{aligned} \|J_3\|_{W^{1,l}(\Omega)} \leq & C\|(A_{d_u} + 1)^{\vartheta_3} J_3\|_{L^p(\Omega)} \\ \leq & C \int_0^t \|(A_{d_u} + 1)^{\vartheta_3} e^{-(t-s)(A_{d_u}+1)} \Phi(u, v)\|_{L^p(\Omega)} \\ \leq & C \int_0^t (t-s)^{-\vartheta_3} e^{-\iota(t-s)} \|uf(u) + u - \gamma ug(v)\|_{L^p(\Omega)} \\ \leq & C \int_0^t (t-s)^{-\vartheta_3} e^{-\iota(t-s)} \|u\|_{L^p(\Omega)} \\ \leq & C \int_0^\infty \sigma^{-\vartheta_3} e^{-\iota\sigma} \\ \leq & C\Gamma(1 - \vartheta_3), \end{aligned}$$

where $\Gamma(1 - \vartheta_3) > 0$ because of $1 - \vartheta_3 > 0$. For $p > 2$, from the Sobolev embedding theorem, we have

$$\|J_3\|_{L^\infty} \leq C\Gamma(1 - \vartheta_3), \text{ for all } t \in (\tau, T_{\max}). \tag{4.5}$$

Therefore, we obtain that $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ is bounded for $t \in (\tau, T_{\max})$ by (4.1), (4.4) and (4.5). ☐

Proof of Theorem 1.1. The assertion of Theorem 1.1 can be obtained by Lemma 4.1(Case 1) and Lemma 2.1 for $\alpha \in (0,1)$. Similarly, Theorem 1.1 can be proved immediately using Lemma 4.1(Case 2) and Lemma 2.1 for $\alpha \geq 1$. ☐

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