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Tur´an type inequalities for the supertrigonometric functions

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Abstract

This paper is devoted to the study of Tur\'{an} type inequalities for some well-known special functions such as supersine and supercosine which are derived by using a new form of the Cauchy-Bunyakovsky-Schwarz inequality.

Turán type inequalities for the supertrigonometric functions

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Abstract: This paper is devoted to the study of Turán type inequalities for some well-known special functions such as supersine and supercosine which are derived by using a new form of the Cauchy-Bunyakovsky-Schwarz inequality.

Key words: Turán type inequality; Supersine; Supercosine; Gauss hypergeometric function; Cauchy-Bunyakovsky-Schwarz inequality.

Mathematics Subject Classification: 33C05; 33C20; 33C99.

Introduction

The importance, in many fields of mathematics, of the inequalities of the type proved by Turán [1]

$$f_n(x)f_{n+2}(x) - [f_{n+1}(x)]^2 \leq 0, \quad n = 0, 1, 2, \cdots, \tag{1}$$

is the well known as Turán type inequalities[2].

Many Turán type inequalities have been investigated in the literature. For example, Joshi and Bissu [3] presented some two-sided inequalities for the ratio of modified Bessel functions of the first kind in 1991. Recently, Segura [4] introduced the bounds for ratios of modified Bessel functions associated with Turán type inequalities in 2011. Baricz [5] recommended Turán type inequalities for q-hypergeometric functions in 2013.

Throughout this paper, let R and N be the sets of the real and integral numbers. The discrete version of the well-known Cauchy-Schwarz inequality [6, 7]

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \quad a_i, b_i \in R, \tag{2}$$

and its integral representation in the space of continuous real-valued functions C ([a,b], R), i.e. the Cauchy-Bunyakovsky-Schwarz (CBS) inequality [6, 7]

$$(\int_a^b [u(t)]^{\frac{1}{2}} [v(t)]^{\frac{1}{2}} dt)^2 \leq (\int_a^b u(t)dt)(\int_a^b v(t)dt), \tag{3}$$

plays an important role in the different branches of modern mathematics.

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In 2006, Laforgia and Natalini [8] used the following form of the CBS inequality:

$$(\int_a^b g(t)[f(t)]^{\frac{m+n}{2}}dt)^2 \leq (\int_a^b g(t)[f(t)]^m dt)(\int_a^b g(t)[f(t)]^n dt), \tag{4}$$

to establish some new Turán type inequalities involving the special functions [9–12] as gamma, polygamma and Riemann's zeta function. Here f and g are the non-negative functions of a real variable and m, $n \in R$, R is the set of real numbers, such that the involved integrals in (1) exist.

In 2018, Bhandari and Bissu presented a new form of the generalized CBS inequality [13]

$$(\int_a^b [u(t)]^\eta [v(t)]^\eta [r(t)]^\eta dt)^2 \leq (\int_a^b [u(t)]^{\eta-l}[v(t)]^{\eta-m}[r(t)]^{\eta-n}dt)(\int_a^b [u(t)]^{\eta+l}[v(t)]^{\eta+m}[r(t)]^{\eta+n}dt), \tag{5}$$

in which η, l, m, $n \in R$, u, v, and r are real integrable functions such that the involved integrals in (5) exist.

In 2021, the supersine via Gauss hypergeometric series, proposed by author [14], was defined as

$$\begin{aligned} {}_2Supersin_1(a,b;c;x) &= \sum_{n=0}^{\infty} \frac{(a)_{2n+1}(b)_{2n+1}}{(c)_{2n+1}} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \frac{\Gamma(c)}{2i\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}[(1-xt)^{-a} - (1+xt)^{-a}]dt, \end{aligned} \tag{6}$$

where a, b, c, $x \in R$, $n \in N$, and $|x| < 1$.

Meanwhile, the supercosine via Gauss hypergeometric series, proposed by author [14], was defined by

$$\begin{aligned} {}_2Supercos_1(a,b;c;x) &= \sum_{n=0}^{\infty} \frac{(a)_{2n}(b)_{2n}}{(c)_{2n}} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1Supercos_0(a;-;xt)dt, \end{aligned} \tag{7}$$

where a, b, c, $x \in R$, $n \in N$, and $|x| < 1$.

The aim of this paper is to study Turán type inequalities for the supersine and supercosine by using a new form of the generalized CBS inequality.

The structure of the paper is as follows: In Section 2, some of main results about Turán type inequality for the supersine and supercosine are obtained and proved in detail, In Section 3, we make a summary and prospect of this paper.

Main results

In this section, we will focus on Turán type inequalities for the supersine and supercosine.

Theorem 1. *Let $b > |\alpha|$, $c - b > |\beta|$ and $|x| \leq 1$. Then Turán type inequality for the supersine is given as*

$$\begin{aligned} [{}_2Supersin_1(a,b;c;x)]^2 \leq &\frac{B(b-\alpha, c-b-\beta)B(b+\alpha, c-b+\beta)}{[B(b,c-b)]^2} \\ &\times {}_2Supersin_1(a-\gamma, b-\alpha; c-(\alpha+\beta); x) \\ &\times {}_2Supersin_1(a+\gamma, b+\alpha; c+(\alpha+\beta); x). \end{aligned} \tag{8}$$

Proof. In the beginning, for (6), let $u(t) = t^{b-1}, v(t) = (1-t)^{c-b-1}$ and $r(t) = (1-xt)^{-a} - (1+xt)^{-a}$. Then the form of the supersine is as follows

$$\begin{aligned}\int_0^1 u(t)v(t)r(t)dt &= \int_0^1 t^{b-1}(1-t)^{c-b-1}[(1-xt)^{-a} - (1+xt)^{-a}]dt \\ &= \frac{2i\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2Supersin_1(a, b; c; x).\end{aligned} \tag{9}$$

On the basis of the inequality (5), we can receive

$$\begin{aligned}&(\int_0^1 t^{\eta(b-1)}(1-t)^{\eta(c-b-1)}[(1-xt)^{-a} - (1+xt)^{-a}]^\eta dt)^2 \\ &\le \int_0^1 t^{(\eta-l)(b-1)}(1-t)^{(\eta-m)(c-b-1)}[(1-xt)^{-a} - (1+xt)^{-a}]^{\eta-n}dt \\ &\times \int_0^1 t^{(\eta+l)(b-1)}(1-t)^{(\eta+m)(c-b-1)}[(1-xt)^{-a} - (1+xt)^{-a}]^{\eta+n}dt.\end{aligned} \tag{10}$$

Consequently, the following results can be obtained by applying (9) and (10),

$$\begin{aligned}&[{}_2Supersin_1(\eta a, \eta(b-1)+1; \eta(c-2)+2; x)]^2 \\ &\le \frac{[\Gamma\{\eta(c-2)+2\}]^2}{[\Gamma\{\eta(b-1)+1\}]^2[\Gamma\{\eta(c-b-1)+1\}]^2} \\ &\times \frac{\Gamma[(\eta-l)(b-1)+1]\Gamma[(\eta-m)(c-b-1)+1]}{\Gamma[(\eta-l)(b-1)+(\eta-m)(c-b-1)+2]} \\ &\times \frac{\Gamma[(\eta+l)(b-1)+1]\Gamma[(\eta+m)(c-b-1)+1]}{\Gamma[(\eta+l)(b-1)+(\eta+m)(c-b-1)+2]} \\ &\times {}_2Supersin_1((\eta-n)a, (\eta-l)(b-1)+1; (\eta-l)(b-1)+(\eta-m)(c-b-1)+2; x) \\ &\times {}_2Supersin_1((\eta+n)a, (\eta+l)(b-1)+1; (\eta+l)(b-1)+(\eta+m)(c-b-1)+2; x).\end{aligned} \tag{11}$$

If $p_1 = \eta(b-1)+1$, $p_2 = \eta(c-b-1)+1$, $p_3 = \eta a$, $q_1 = l(b-1)$, $q_2 = m(c-b-1)$, and $q_3 = na$ in inequality (11), then Turán type inequality for the supersine can be written as

$$\begin{aligned}[{}_2Supersin_1(p_3, p_1; p_1+p_2; x)]^2 \le &\frac{[\Gamma(p_1+p_2)]^2}{[\Gamma(p_1)]^2[\Gamma(p_2)]^2}\frac{\Gamma(p_1-q_1)\Gamma(p_2-q_2)}{\Gamma[(p_1-q_1)+(p_2-q_2)]}\frac{\Gamma(p_1+q_1)\Gamma(p_2+q_2)}{\Gamma[(p_1+q_1)+(p_2+q_2)]} \\ &\times {}_2Supersin_1(p_3-q_3, p_1-q_1; (p_1-q_1)+(p_2-q_2); x) \\ &\times {}_2Supersin_1(p_3+q_3, p_1+q_1; (p_1+q_1)+(p_2+q_2); x).\end{aligned} \tag{12}$$

Using the relationship between beta and gamma functions

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m, n > 0, \tag{13}$$

Turán type inequality can also be shown in the following form

$$\begin{aligned}[{}_2Supersin_1(p_3, p_1; p_1+p_2; x)]^2 \le &\frac{B(p_1-q_1, p_2-q_2)B(p_1+q_1, p_2+q_2)}{[B(p_1, p_2)]^2} \\ &\times {}_2Supersin_1(p_3-q_3, p_1-q_1; (p_1-q_1)+(p_2-q_2); x) \\ &\times {}_2Supersin_1(p_3+q_3, p_1+q_1; (p_1+q_1)+(p_2+q_2); x),\end{aligned} \tag{14}$$

where $p_1 > |q_1|$, $p_2 > |q_2|$ and $|x| \leq 1$.

When $\eta = 1$, the inequality (11) reduces to following inequality for the supersine

$$\begin{aligned}
[\,{}_2Supersin_1(a,b;c;x)]^2 \leq \; & \frac{[\Gamma(c)]^2}{[\Gamma(b)]^2[\Gamma(c-b)]^2} \\
& \times \frac{\Gamma[b-l(b-1)]\Gamma[(c-b)-m(c-b-1)]}{\Gamma[c-l(b-1)-m(c-b-1)]} \\
& \times \frac{\Gamma[b+l(b-1)]\Gamma[(c-b)+m(c-b-1)]}{\Gamma[c+l(b-1)+m(c-b-1)]} \\
& \times \, {}_2Supersin_1((1-n)a, b-l(b-1); c-l(b-1)-m(c-b-1); x) \\
& \times \, {}_2Supersin_1((1+n)a, b+l(b-1); c+l(b-1)+m(c-b-1); x).
\end{aligned} \tag{15}$$

Suppose $l(b-1) = \alpha$, $m(c-b-1) = \beta$, $na = \gamma$, $b > |\alpha|$, $c-b > |\beta|$ and $|x| \leq 1$. Then, through (13) the inequality (15) transforms to the much nicer version of Turán type inequality for the supersine as follows

$$\begin{aligned}
[\,{}_2Supersin_1(a,b;c;x)]^2 \leq \; & \frac{B(b-\alpha, c-b-\beta)B(b+\alpha, c-b+\beta)}{[B(b,c-b)]^2} \\
& \times \, {}_2Supersin_1(a-\gamma, b-\alpha; c-(\alpha+\beta); x) \\
& \times \, {}_2Supersin_1(a+\gamma, b+\alpha; c+(\alpha+\beta); x).
\end{aligned} \tag{16}$$

Theorem 2. *For $b > |\lambda|$, $c-b > |\mu|$ and $|x| \leq 1$, we have Turán type inequality for the supercosine*

$$\begin{aligned}
[\,{}_2Supercos_1(a,b;c;x)]^2 \leq \; & \frac{B(b-\lambda, c-b-\mu)B(b+\lambda, c-b+\mu)}{[B(b,c-b)]^2} \\
& \times \, {}_2Supercos_1(a-\omega, b-\lambda; c-(\lambda+\mu); x) \\
& \times \, {}_2Supercos_1(a+\omega, b+\lambda; c+(\lambda+\mu); x).
\end{aligned} \tag{17}$$

Proof . At first, basing on (6), if $u(t) = t^{b-1}$, $v(t) = (1-t)^{c-b-1}$, and $r(t) = \,{}_1Supercos_0(a;-;xt)$, we have

$$\begin{aligned}
\int_0^1 u(t)v(t)r(t)dt &= \int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1Supercos_0(a;-;xt)dt \\
&= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2Supercos_1(a,b;c;x).
\end{aligned} \tag{18}$$

By the generalized CBS inequality, the following inequality for the supercosine can be written as

$$\begin{aligned}
& \left(\int_0^1 t^{\eta(b-1)}(1-t)^{\eta(c-b-1)}[\,{}_1Supercos_0(a;-;xt)]^{\eta}\right)^2 \\
& \leq \int_0^1 t^{(\eta-l)(b-1)}(1-t)^{(\eta-m)(c-b-1)}[\,{}_1Supercos_0(a;-;xt)]^{\eta-n}dt \\
& \times \int_0^1 t^{(\eta+l)(b-1)}(1-t)^{(\eta+m)(c-b-1)}[\,{}_1Supercos_0(a;-;xt)]^{\eta+n}dt.
\end{aligned} \tag{19}$$

Thus, the following results can be obtained by applying (18) and (19),

$$\begin{aligned}
&[{}_2Supercos_1(\eta a, \eta(b-1)+1; \eta(c-2)+2; x)]^2 \\
&\le \frac{[\Gamma\{\eta(c-2)+2\}]^2}{[\Gamma\{\eta(b-1)+1\}]^2[\Gamma\{\eta(c-b-1)+1\}]^2} \\
&\times \frac{\Gamma[(\eta-l)(b-1)+1]\Gamma[(\eta-m)(c-b-1)+1]}{\Gamma[(\eta-l)(b-1)+(\eta-m)(c-b-1)+2]} \\
&\times \frac{\Gamma[(\eta+l)(b-1)+1]\Gamma[(\eta+m)(c-b-1)+1]}{\Gamma[(\eta+l)(b-1)+(\eta+m)(c-b-1)+2]} \\
&\times {}_2Supercos_1((\eta-n)a, (\eta-l)(b-1)+1; (\eta-l)(b-1)+(\eta-m)(c-b-1)+2; x) \\
&\times {}_2Supercos_1((\eta+n)a, (\eta+l)(b-1)+1; (\eta+l)(b-1)+(\eta+m)(c-b-1)+2; x).
\end{aligned} \tag{20}$$

Next, let $p_1 = \eta(b-1)+1$, $p_2 = \eta(c-b-1)+1$, $p_3 = \eta a$, $q_1 = l(b-1)$, $q_2 = m(c-b-1)$, and $q_3 = na$ in inequality (20). Then we can receive Turán type inequality for the supercosine as follows:

$$\begin{aligned}
[{}_2Supercos_1(p_3, p_1; p_1+p_2; x)]^2 \le{}& \frac{[\Gamma(p_1+p_2)]^2}{[\Gamma(p_1)]^2[\Gamma(p_2)]^2} \frac{\Gamma(p_1-q_1)\Gamma(p_2-q_2)}{\Gamma[(p_1-q_1)+(p_2-q_2)]} \frac{\Gamma(p_1+q_1)\Gamma(p_2+q_2)}{\Gamma[(p_1+q_1)+(p_2+q_2)]} \\
&\times {}_2Supercos_1(p_3-q_3, p_1-q_1; (p_1-q_1)+(p_2-q_2); x) \\
&\times {}_2Supercos_1(p_3+q_3, p_1+q_1; (p_1+q_1)+(p_2+q_2); x).
\end{aligned} \tag{21}$$

On the other hand, for $p_1 > |q_1|$, $p_2 > |q_2|$ and $|x| \le 1$, Turán type inequality for the supercosine can also be expressed in the following form by applying (13),

$$\begin{aligned}
[{}_2Supercos_1(p_3, p_1; p_1+p_2; x)]^2 \le{}& \frac{B(p_1-q_1, p_2-q_2)B(p_1+q_1, p_2+q_2)}{[B(p_1, p_2)]^2} \\
&\times {}_2Supercos_1(p_3-q_3, p_1-q_1; (p_1-q_1)+(p_2-q_2); x) \\
&\times {}_2Supercos_1(p_3+q_3, p_1+q_1; (p_1+q_1)+(p_2+q_2); x).
\end{aligned} \tag{22}$$

After that, suppose $\eta = 1$. Then inequality (20) can be simplified to the following inequality

$$\begin{aligned}
[{}_2Supercos_1(a, b; c; x)]^2 \le{}& \frac{[\Gamma(c)]^2}{[\Gamma(b)]^2[\Gamma(c-b)]^2} \\
&\times \frac{\Gamma[b-l(b-1)]\Gamma[(c-b)-m(c-b-1)]}{\Gamma[c-l(b-1)-m(c-b-1)]} \\
&\times \frac{\Gamma[b+l(b-1)]\Gamma[(c-b)+m(c-b-1)]}{\Gamma[c+l(b-1)+m(c-b-1)]} \\
&\times {}_2Supercos_1((1-n)a, b-l(b-1); c-l(b-1)-m(c-b-1); x) \\
&\times {}_2Supercos_1((1+n)a, b+l(b-1); c+l(b-1)+m(c-b-1); x).
\end{aligned} \tag{23}$$

Finally, for $l(b-1) = \lambda$, $m(c-b-1) = \mu$, $na = \omega$, $b > |\lambda|$, $c-b > |\mu|$ and $|x| \le 1$, by using (13) Turán type inequality (23) for the supercosine can be further simplified to

$$\begin{aligned}
[{}_2Supercos_1(a, b; c; x)]^2 \le{}& \frac{B(b-\lambda, c-b-\mu)B(b+\lambda, c-b+\mu)}{[B(b, c-b)]^2} \\
&\times {}_2Supercos_1(a-\omega, b-\lambda; c-(\lambda+\mu); x) \\
&\times {}_2Supercos_1(a+\omega, b+\lambda; c+(\lambda+\mu); x).
\end{aligned} \tag{24}$$

Conclusion

In our work we had addressed the theorems about Turán type inequalities for the supersine and supercosine and proved it by using a new form of the generalized CBS inequality. This paper will strongly promote the rapid development of the field of special functions.

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