Analytical solution of a fractional differential equation in the theory of viscoelastic fluids

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Abstract

The aim of this paper is to present analytical solutions of fractional delay differential equations (FDDEs) of an incompressible generalized Oldroyd-B fluid with fractional derivatives of Caputo type. Using a modification of the method of separation of variables the main equation with non-homogeneous boundary conditions is transformed into an equation with homogeneous boundary conditions, and the resulting solutions are then expressed in terms of Green functions via Laplace transforms.This results presented in two condition , in first step when $0 \leq \alpha, \beta \leq \frac{1}{2}$ and in the second step we considered $\frac{1}{2} \leq \alpha, \beta \leq 1$,for each step 1,2 for the unsteady flows of a generalized Oldroyd-B fluid, including a flow with a moving plate, are considered via examples.

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Introduction

Many real-world processes can be cast generally in the form of fractional differential systems with integer order (i.e., ordinary differential equations and systems) but there is a growing number of researchers that believe that fractional-differential equations can describe and model and complex physical processes more accurately than the corresponding ordinary differential equations. So, in recent decades, the search for analytical and numerical solutions to fractional differential equations has been of considerable interest (Podlubny, 1998; Diethelm & Freed, 1999; Magin, 2006; Kilbas et al., 2006). Fractional differential equations can be applied to the dynamic modeling of non-Newtonian fluids: for example, in the modeling of melting plastics and in the study of emulsion plastics or soft tissue. Practically speaking there are few Newtonian fluids in reality, so most fluids are of the non-Newtonian type which there is no linear relationship between the stress tensor and the deformation tensor (Fetecau et al., 2008).

Viscoelastic fluids form are an important class of non-Newtonian fluids which exhibit both elastic and viscous properties. Among them the so-called *Oldroyd-B* fluid can be used to describe the response of fluids that have a small memory. This means that whenever they flow, these fluids will spend less time to find the first state and stability (Khan, 2009; Nadeem, 2007). Due to the wide range of applications of these fluids, considerable attention has been paid to the prediction of the behavior of non-Newtonian fluids. Structural equations that are presented in a constitutive rheological fashion have a fractional calculation, so they are very effective for working with viscoelastic properties (Song & Jiang, 1998; Hilfer, 2000). The viscoelastic fluid equations in fractional models are obtained by replacing ordinary derivatives with one of many possible definitions of fractional derivatives in the defining equations. In the study of fluids we deal with a phenomenon called delay which is due to the distance between the sensor and the source of changes arising from e.g., plumbing, measurement slowness, or complex dynamics. Different methods for finding analytical solutions of these type of equations are proposed: An analytical solution for unsteady helical flows is presented by Dang *et al* in (Tong et al., 2009). In Haitao and Mingyu (Haitao & Mingyu, 2009) there is a discussion of an Oldroyd-B fluid between two parallel plates. In addition, Fetecau (Fetecau et al., 2009; Vieru et al., 2008) developed a generalization of the flow of viscoelastic fluids between two-sided walls. Then Hyder(Shah et al., 2009), Qi (Qi & Jin, 2009), Zheng *et al* (Zheng et al., 2011) and Hayat (Hayat et al., 2007) discussed the generalized flow of an Oldroyd-B fluid under varying conditions. In closing this brief review we mention that Javidi and saedshoar (Heris & Javidi, 2017), gave analytical solutions of various forms of such delay equations.

Many events in the natural world can be modeled to form of fractional delay differential equations (FDDEs). FDDEs have important applications in many fields for example technology, economics, biology, medical science, physics and finance (Wang et al., 2011). Some numerical methods for FDDEs are introduced in (Heris & Javidi, 2018; Morgado et al., 2013; Čermák et al., 2016; Lazarević & Spasić, 2009) and etc . Saedshoar and javidi (Heris & Javidi, 2017), proposed a numerical method based on fractional backward differential formulas (FBDF) for solving fractional delay differential equations. Also they find the Green's functions for this equation corresponding to periodic/ anti-periodic conditions in terms of the functions of Mittag Leffler type.

In this paper we present analytical solutions for unsteady flows of a generalized Oldroyd-B fluid with constant delay time using Riemann-Liouville fractional derivatives as the defining derivatives. A new separation of variables method (Jiang et al., 2012) and use of Laplace transforms for the Riemann-Liouville fractional derivative are adapted to solve the new governing equation for fractional differential equations with constant delay when applied to viscoelastic fluids.
The paper is structured as follows: In Section 2, we recall some basic definitions of fractional calculus. In section 3 we give the derivation of the governing equation. Section 4 deals with the method of separation of variables, the Laplace transformation applied to fractional derivatives in two steps $0 \le \alpha, \beta \le \frac{1}{2}$ and $\frac{1}{2} \le \alpha, \beta \le 1$, and the method of solution for each two steps separatively. Finally, in section 5 we give the examples dealing with varying initial conditions by consider two condition for α and β.

Preliminaries

In this section, we will introduce some of the fundamental definitions.

Definition 2.1 ((Podlubny, 1998)). Euler's gamma function is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \;\; Re(z) > 0.$$

(1)

$C(J,R)$ denotes the Banach space of all continuous functions from $J=[0,T]$ into R with the norm

$$\|u\|_{\infty} = \sup\{|u(t)| : t \in J\}, \quad T > 0.$$

(2)

$C^n(J,R)$ denotes the class of all real valued functions defined on $J=[0,T]$, $T>0$ which have continuous nth order derivatives.

Definition 2.2 ((Kilbas et al., 2006)). The fractional integral of order $\alpha > 0$ of the function $f \in C(J,R)$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad 0 < t < T.$$

(3)

Definition 2.3 ((Kilbas et al., 2006)). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of the function $f \in C(J,R)$ is defined as

$$^{RL}D^{\alpha} f(t) = \{$$

$$D^n I^{n-\alpha} f(t) = \tfrac{1}{\Gamma(n-\alpha)} \left(\tfrac{d^n}{dt^n}\right) \int_0^t \tfrac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$
$$n-1 < \alpha < n, \ n \in N,$$
$$f^{(n)}(t), \qquad \alpha = n.$$

(4)

Definition 2.4 ((Kilbas et al., 2006)). The Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C^n(J,R)$ is defined as

$$^{C}D^{\alpha} f(t) = \{$$

$$I^{n-\alpha} D^n f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_0^t \tfrac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$
$$n-1 < \alpha < n, \ n \in N,$$
$$f^{(n)}(t), \qquad \alpha = n.$$

(5)

Definition 2.5 ((Kilbas et al., 2006)). Mittag-leffler functions defined by

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}\ ,\ \ x, \beta \in C, Re(\alpha) > 0, E_\alpha(x) = E_{\alpha,1}.$$

(6)

Definition 2.6 ((Čermák et al., 2016)). The generalized delay exponential function (of Mittag–Leffler type) is given by

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty} \binom{j+m}{j} \frac{\lambda^j (t-(m+j)\tau)^{\alpha(m+j)+\beta-1}}{\Gamma(\alpha(m+j)+\beta)} H(t-(m+j)\tau), t > 0,$$

(7)

where $\lambda \in C$, $\alpha, \beta, \tau \in R$ and $m \in Z$ and $H(z)$ is the Heaviside step function. If $\lambda \in C$, $\alpha, \beta, \tau \in R$ and $m \in Z$ then laplace transform of $G_{\alpha,\beta}^{\lambda,\tau,m}(t)$ is:

$$L(G_{\alpha,\beta}^{\lambda,\tau,m}(t))(s) = \frac{s^{\alpha-\beta}\exp\{-ms\tau\}}{(s^\alpha - \lambda\exp\{-s\tau\})^{m+1}}, \quad s > 0.$$

(8)

Governing equations

The fundamental equations governing the unsteady motion of an incompressible fluid are

$$\operatorname{div} V = 0,$$

(9)

$$\rho \frac{dV}{dt} = -\nabla p + \operatorname{div} S + F_b.$$

(10)

The constitutive equation for a generalized Oldroyd-B fluid is given by (Qi & Jin, 2009; Zheng et al., 2011),

$$(1 + \lambda^{\alpha} \frac{D^{\alpha}}{Dt^{\alpha}})S = \mu(1 + \lambda^{\beta} \frac{D^{\beta}}{Dt^{\beta}})A_1 \ ,$$

(11)

where $V = (u, v, w)$ is the fluid velocity, $S = (S_{i,j})$ is the extra-stress tensor, $A_1 = (\nabla V) + (\nabla V)^T$ present the first Rivlin-Ericksen tensor, ∇ is the gradient operator, and p is the pressure. Here $F_b = (F_{bx}, F_{by}, F_{bz})$ is the body force, ρ, μ are the density and the dynamic viscosity coefficient of the fluid respectively, λ_α and λ_β are the material constants that represent the relaxation time and retardation time, respectively, and α, β denote the orders of the fractional derivatives, i.e., real numbers that satisfy $0 \leq \alpha, \beta \leq 1$. Furthermore, $\frac{D^\alpha}{Dt^\alpha}$ and $\frac{D^\beta}{Dt^\beta}$ are fractional material derivatives that can be expressed as

$$\frac{D^{\alpha} S}{Dt^{\alpha}} = D_t^{\alpha} S + (V.\nabla)S - (\nabla.V)S - S(\nabla V)^T,$$

(12)

$$\frac{D^{\beta} S}{Dt^{\beta}} = D_t^{\beta} S + (V.\nabla)S - (\nabla.V)S - S(\nabla V)^T.$$

(13)

In Eq. (11), (13), the fractional derivative operator D^α is taken in the Caputo.

We consider unidirectional flow, that is the case where the velocity and the stress take the form where i is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. Using Eq. (14) below, the continuity Eq. (9) is satisfied automatically while Eq. (12), bearing in mind the initial condition $S(y,0)=0$, leads to the following relationships for the constitutive equation

$$S_{xz}=S_{zy}=S_{yz}=S_{zz}=S_{yy}=0, \qquad S_{yx}=S_{xy}, S_{zx}=S_{xz},$$

$$\left(1+\lambda_\alpha D_t^\alpha\right) S_{xy} = \mu\left(1+\lambda_\beta D_t^\beta\right)\frac{\partial u}{\partial y}\ ,$$

(14)

$$\left(1+\lambda_\alpha D_t^\alpha\right) S_{xx} - 2\lambda_\alpha S_{xy}\frac{\partial u}{\partial y} = -2\mu\lambda_\beta\left(\frac{\partial u}{\partial y}\right)^2 .$$

Substituting Eqs.(15) in to momentum equation (10), we have thebfollowing equation in x-direction:

$$\left(1+\lambda_\alpha D_t^\alpha\right)\frac{\partial u}{\partial t} = v\left(1+\lambda_\beta D_t^\beta\right)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\left(1+\lambda_\alpha D_t^\alpha\right)\left(F_{bx}-\frac{\partial p}{\partial x}\right)\ .$$

(15)

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity coefficient of fluid.
The constitutive equation of a generalized Burgers fluid is

$$\left(1+\lambda_\alpha\frac{D^\alpha}{Dt^\alpha}+\theta\frac{D^{2\alpha}}{Dt^{2\alpha}}\right)\ S = \mu\left(1+\lambda_\beta\frac{D^\beta}{Dt^\beta}\right)A_1\ , \quad (0<\alpha\ ,\beta\le 1)\,,$$

(16)

where θ is the material constant.
combining the constitutive equation (17) with the equation (10) , we get the following fractional Burgers fluid model

$$\left(1+\lambda_\alpha D_t^\alpha+\theta D_t^{2\alpha}\right)\frac{\partial u}{\partial t} = v\left(1+\lambda_\beta D_t^\beta\right)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\left(1+\lambda_\alpha D_t^\alpha+\theta D_t^{2\alpha}\right)\left(F_{bx}-\frac{\partial p}{\partial x}\right)$$

(17)

Where $\nu = {}^{\mu}/_{\rho}$. Eqs.(16)and (18) have the following form

$$a_0 {D_t}^{2\alpha+1} u(y,t) + a_1 {D_t}^{\alpha+1} u(y,t) + a_2 {D_t}^{2\alpha} u(y,t) + a_3 {D_t}^{1} u(y,t)$$

(18)

$$+a_4 {D_t}^{\alpha} u(y,t) + a_5 u(y,t) = b_1 D_t^{\beta} \frac{\partial^2 u(y,t)}{\partial y^2} + b_2 \frac{\partial^2 u(y,t)}{\partial y^2} + \bar{f}\left(y,t\right),$$

the delay form of Eqs(19)is

$$a_0 {D_t}^{2\alpha+1} u(y,t) + a_1 {D_t}^{\alpha+1} u(y,t) + a_2 {D_t}^{2\alpha} u(y,t) + a_3 {D_t}^{1} u(y,t)$$

(19)

$$+a_4 {D_t}^{\alpha} u(y,t) + a_5 u(y,t-\tau) = b_1 D_t^{\beta} \frac{\partial^2 u(y,t)}{\partial y^2} + b_2 \frac{\partial^2 u(y,t)}{\partial y^2} + \bar{f}\left(y,t\right).$$

The associated initial and boundary conditions are as follows:

$$u\left(y,t\right) = \psi_1\left(y,t\right), \qquad u\left(0,t\right) = \varphi_1\left(t\right), -\tau \le t \le 0,$$

$$u_t\left(y,t\right) = \psi_2\left(y,t\right), \qquad u\left(L,t\right) = \varphi_1\left(t\right), \qquad 0 < \alpha, \beta < 1.$$

A method of separation of variables

At first, the problem involves non-homogeneous boundary conditions. We want transform it into a problem with homogeneous boundary conditions. So, consider

$$u\left(y,t\right) = W\left(y,t\right) + V\left(y,t\right),$$

(20)

where

$$V\left(y,t\right) = \left(1 - \frac{y}{L}\right)\varphi_1\left(t\right) + \frac{y}{L}\varphi_2\left(t\right),$$

(21)

which satisfies the boundary conditions

$$V\left(0,t\right) = \varphi_1\left(t\right), V\left(L,t\right) = \varphi_2\left(t\right).$$

Using Eqs.(20) and (21) along with the associated initial and boundary conditions above, we have

$W\left(y,t\right)+\left(1-\frac{y}{L}\right)\varphi_1\left(t\right)+\frac{y}{L}\varphi_2\left(t\right)=\psi_1\left(y,t\right), \qquad -\tau\le t\le 0,$
$W_t\left(y,t\right)+\left(1-\frac{y}{L}\right){\varphi'}_1\left(t\right)+\frac{y}{L}{\varphi'}_2\left(t\right)=\psi_2\left(y,t\right),$
$W\left(L,t\right)+V\left(L,t\right)=\varphi_2\left(t\right),$
$W\left(L,t\right)+V\left(L,t\right)=\varphi_2(t),$
$W\left(y,t\right)=\psi_1\left(y,t\right)-\left(1-\frac{y}{L}\right)\varphi_1\left(t\right)-\frac{y}{L}\varphi_2\left(t\right)=\overline{\psi_1}\left(y,t\right),$
$W_t\left(y,t\right)=\psi_1\left(y,t\right)-\left(1-\frac{y}{L}\right){\varphi'}_1\left(t\right)-\frac{y}{L}{\varphi'}_2\left(t\right)=\overline{\psi_2}(y,t).$

Now main problem is solving

$$a_0{D_t}^{2\alpha+1}W\left(y,t\right)+a_1{D_t}^{\alpha+1}W\left(y,t\right)+a_2{D_t}^{2\alpha}W\left(y,t\right)+a_3{D_t}^{1}W\left(y,t\right)$$

(22)

$+a_4{D_t}^{\alpha}W\left(y,t\right)+a_5W\left(y,t-\tau\right)-b_1D_t^{\beta}\frac{\partial^2 w(y,t)}{\partial y^2}-b_2\frac{\partial^2 w(y,t)}{\partial y^2}$

$=-a_0{D_t}^{2\alpha+1}V\left(y,t\right)-a_1{D_t}^{\alpha+1}V\left(y,t\right)-a_2{D_t}^{2\alpha}V\left(y,t\right)-a_3{D_t}^{1}V\left(y,t\right)$

where the initial condition is

$\sum\limits_{n=1}^{\infty}B_n\left(0\right)\sin\frac{n\pi y}{L}=\sum\limits_{n=1}^{\infty}d_n^{(1)}\left(0\right)\sin\frac{n\pi y}{L}-\sum\limits_{n=1}^{\infty}\frac{2}{n\pi}\left[\varphi_1\left(0\right)-\left(-1\right)^n\varphi_2\left(0\right)\right]\sin\frac{n\pi y}{L},$

$\sum\limits_{n=1}^{\infty}{B'}_n\left(0\right)\sin\frac{n\pi y}{L}=\sum\limits_{n=1}^{\infty}d_n^{(2)}\left(0\right)\sin\frac{n\pi y}{L}-\sum\limits_{n=1}^{\infty}\frac{2}{n\pi}\left[{\varphi'}_1\left(0\right)-\left(-1\right)^n{\varphi'}_2\left(0\right)\right]\sin\frac{n\pi y}{L},$

and

$d_n^{(i)}=\frac{2}{L}\int\limits_0^L\bar{\psi}_i\ \left(y,0\right)\ sin\frac{n\pi y}{L}dy, \qquad i=1,2.$

Let

$$W\left(y,t\right)=\sum_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L},$$

$$\overline{\psi_i}\ \left(y\right)=\sum_{n=1}^{\infty}d_n^{(i)}\sin\frac{n\pi y}{L}\quad\left(i=1,2,\ldots,m\right).$$

Then, we have

$a_0{D_t}^{2\alpha+1}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}+a_1{D_t}^{\alpha+1}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}+a_2{D_t}^{2\alpha}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}$
$+a_3{D_t}^{1}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}+a_4{D_t}^{\alpha}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}+a_5\sum\limits_{n=1}^{\infty}B_n\left(t-\tau\right)\sin\frac{n\pi y}{L}$
$-b_1\left(\frac{n\pi}{L}\right)^2D_t^{\beta}\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}-b_2\left(\frac{n\pi}{L}\right)^2\sum\limits_{n=1}^{\infty}B_n\left(t\right)\sin\frac{n\pi y}{L}$
$=-a_0\frac{2}{n\pi}{D_t}^{2\alpha+1}\sum\limits_{n=1}^{\infty}\left[\varphi_1\left(t\right)-\left(-1\right)^n\varphi_2\left(t\right)\right]\sin\frac{n\pi y}{L}-a_1\frac{2}{n\pi}{D_t}^{\alpha+1}\sum\limits_{n=1}^{\infty}\left[\varphi_1\left(t\right)-\left(-1\right)^n\varphi_2\left(t\right)\right]\sin\frac{n\pi y}{L}$

$-a_2\frac{2}{n\pi}{D_t}^{2\alpha}\sum\limits_{n=1}^{\infty}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L}-a_3\frac{2}{n\pi}{D_t}^{1}\sum\limits_{n=1}^{\infty}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L}$
$-a_4\frac{2}{n\pi}{D_t}^{\alpha}\sum\limits_{n=1}^{\infty}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L}-a_5\frac{2}{n\pi}\sum\limits_{n=1}^{\infty}\left[\varphi_1(t-\tau)-(-1)^n\varphi_2(t-\tau)\right]\sin\frac{n\pi y}{L}$
$+\sum\limits_{n=1}^{\infty}f_n(t)\sin\frac{n\pi y}{L}$
Equating coefficients leads to

$$a_0{D_t}^{2\alpha+1}B_n(t)+a_1{D_t}^{\alpha+1}B_n(t)+a_2{D_t}^{2\alpha}B_n(t)+a_3{D_t}^{1}B_n(t)$$

(23)

$$\begin{aligned}&+a_4{D_t}^{\alpha}B_n(t)+a_5B_n(t-\tau)-b_1\left(\frac{n\pi}{L}\right)^2D_t^{\beta}B_n(t)-b_2\left(\frac{n\pi}{L}\right)^2B_n(t)\\&=-a_0\frac{2}{n\pi}{D_t}^{2\alpha+1}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]-a_1\frac{2}{n\pi}{D_t}^{\alpha+1}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]\\&-a_2\frac{2}{n\pi}{D_t}^{2\alpha}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]-a_3\frac{2}{n\pi}{D_t}^{1}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]\\&-a_4\frac{2}{n\pi}{D_t}^{\alpha}\left[\varphi_1(t)-(-1)^n\varphi_2(t)\right]-a_5\frac{2}{n\pi}\left[\varphi_1(t-\tau)-(-1)^n\varphi_2(t-\tau)\right]+f_n(t).\end{aligned}$$

with the boundary conditions

$$\begin{aligned}B_n(0)&=d_n^{(1)}(0)-\frac{2}{n\pi}\varphi_1(0)+(-1)^n\frac{2}{n\pi}\varphi_2(0),\\B'_n(0)&=d_n^{(2)}(0)-\frac{2}{n\pi}\varphi'_1(0)+(-1)^n\frac{2}{n\pi}\varphi'_2(0).\end{aligned}$$

In this part we divide the main problem in two part

$0\le\alpha,\beta\le\frac{1}{2}$

: when and $\frac{1}{2}\le\alpha,\beta\le 1$

Applying the Laplace transform with respect to t defined by

$$\bar{B}_n(s)=\int\limits_0^{\infty}e^{-st}B_n(t)\,dt.$$

in Eq. (23), we obtain

$a_0s^{2\alpha+1}\bar{B}_n(s)-a_0s^{2\alpha}B_n(0)+a_1s^{\alpha+1}\bar{B}_n(s)-a_1s^{\alpha}B_n(0)+a_2s^{2\alpha}\bar{B}_n(s)-a_2s^{2\alpha-1}B_n(0)$

$+a_3s\bar{B}_n(s)-a_3B_n(0)+a_4s^{\alpha}\bar{B}_n(s)-a_4s^{\alpha-1}B_n(0)+a_5e^{-s\tau}\left[\int\limits_{-\tau}^{0}e^{-sp}B_n(p)\,dp\right]$

$-a_5e^{-s\tau}\bar{B}_n(s) - b_1\left(\frac{n\pi}{L}\right)^2 s^\beta \bar{B}_n(s) + b_1\left(\frac{n\pi}{L}\right)^2 s^{\beta-1}B_n(0) - b_2\left(\frac{n\pi}{L}\right)^2 \bar{B}_n(s)$

$= -a_0\frac{2}{n\pi}s^{2\alpha+1}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_0\frac{2}{n\pi}s^{2\alpha}\left[d_n^{(1)}(0) - B_n(0)\right]$

$-a_1\frac{2}{n\pi}s^{\alpha+1}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_1\frac{2}{n\pi}s^{\alpha}\left[d_n^{(1)}(0) - B_n(0)\right]$

$-a_2\frac{2}{n\pi}s^{2\alpha}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_2\frac{2}{n\pi}s^{2\alpha-1}\left[d_n^{(1)}(0) - B_n(0)\right]$

$-a_3\frac{2}{n\pi}s\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_3\frac{2}{n\pi}\left[d_n^{(1)}(0) - B_n(0)\right]$

$-a_4\frac{2}{n\pi}s^{\alpha}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_4\frac{2}{n\pi}s^{\alpha-1}\left[d_n^{(1)}(0) - B_n(0)\right] - a_5\frac{2}{n\pi}e^{-s\tau}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right]$

$+a_5\frac{2}{n\pi}e^{-s\tau}\left[\int\limits_{-\tau}^{0} e^{-sp}\left[\varphi_1(p) - (-1)^n\varphi_2(p)\right]dp\right] + F_n(s)$

By assumption $H(S) = \int\limits_{-\tau}^{0} e^{-sp}\left[\varphi_1(p) - (-1)^n\varphi_2(p)\right]dp$, $G(s) = \int\limits_{-\tau}^{0} e^{-sp}B_n(p)\,dp$ and $k_n = \frac{n\pi}{L}$, so we can write

$$+\frac{-a_5G(s)e^{-s\tau} + a_5\frac{2}{k_nL}e^{-s\tau}H(S) + F_n(s)}{a_0s^{2\alpha+1} + a_1s^{\alpha+1} + a_2s^{2\alpha} + a_3s + a_4s^\alpha - a_5e^{-s\tau} - b_1{k_n}^2s^\beta - b_2{k_n}^2}$$

(24)

Using Eq.(24) we rewrite Eq.(23) as

$$\{B_n(0)[a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1} - \frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} + a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1} - \frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}$$

$$
\begin{aligned}
\bar{B}_n(s) = e^{sm\tau}\sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\geq 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{{a_0}^{m+1}}\frac{m!\left(-{k_n}^2\right)^{n+q}}{k!i!j!l!q!}{a_1}^k{a_2}^i{a_3}^j{a_4}^l{b_1}^n{b_2}^q \\
+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} \\
+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}] \\
-\frac{2}{k_nL}\left[\overline{\varphi_1}(s)-(-1)^n\overline{\varphi_2}(s)\right][a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} \\
+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} \\
+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}] \\
+\frac{2}{k_nL}\left[d_n^{(1)}(0)-B_n(0)\right][a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} \\
+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}] \\
-a_5G(s)e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5\frac{2}{k_nL}e^{-s\tau}H(S)\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}} \\
+F_n(s)\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}\}
\end{aligned}
$$

Applying the discrete inverse Laplace transform to the preceding equation, we obtain

$$
B_n(t) = \sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\geq 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{{a_0}^{m+1}}\frac{m!\left(-{k_n}^2\right)^{n+q}}{k!i!j!l!q!}{a_1}^k{a_2}^i{a_3}^j{a_4}^l{b_1}^n{b_2}^q
$$

$$
\begin{aligned}
\{B_n(0)H(t-m\tau)[a_0G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+a_1G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau) \\
+a_2G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+a_3G_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau) \\
+a_4G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)-b_1{k_n}^2G_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)] \\
-\frac{2}{k_nL}[\int_0^t\left[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)\right]H(u-m\tau)(a_0G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau)
\end{aligned}
$$

$$+a_1 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n}(u-m\tau)+a_2 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(u-m\tau)$$

$$+a_3 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n}(u-m\tau)+a_4 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(u-m\tau))du]$$

$$-a_5\frac{2}{k_n L}\int_0^t\left[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)\right]H(u-\tau(m+1))$$

$$G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$+\frac{2}{k_n L}\left[d_n^{(1)}(0)-B_n(0)\right][a_0 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(t-m\tau)$$

$$+a_1 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(t-m\tau)+a_2 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)$$

$$+a_3 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(t-m\tau)+a_4 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)]$$

$$-a_5\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$-a_5\frac{2}{k_n L}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$+\int_0^t f_n(t-u)\,H(u-m\tau)\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

Once the $B_n(t)$ are known, so are the $W(y,t)$, and thus $u(y,t)$ as desired.

$\frac{1}{2}\le\alpha,\beta\le 1$

In the same way in the subsection4.1 we could have

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\ge 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{a_0^{m+1}}\frac{m!\left(-k_n{}^2\right)^{n+q}}{k!i!j!l!q!}a_1{}^k a_2{}^i a_3{}^j a_4{}^l b_1{}^n b_2{}^q$$

$$\{B_n(0)\,H(t-m\tau)\,[a_0 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(t-m\tau)+a_1 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(t-m\tau)$$

$$+a_2 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)+a_3 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(t-m\tau)$$

$$+a_4 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)-b_1 k_n{}^2 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+2}(t-m\tau)]$$

$$+B'_n(0)\,H(t-m\tau)\,[a_0 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)+a_1 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)$$

$$+a_2 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)+a_4 G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+3}(t-m\tau)]$$

$$+B''_n(0)\,a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)$$

$$-\frac{2}{k_nL}[\int_0^t[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)]\,H(u-m\tau)\,(a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n}(u-m\tau)$$

$$+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n}(u-m\tau)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(u-m\tau)$$

$$+a_3G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n}(u-m\tau)+a_4G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(u-m\tau))du]$$

$$-a_5\frac{2}{k_nL}\int_0^t[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)]\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$+\frac{2}{k_nL}\left[d_n^{(1)}(0)-B_n(0)\right][a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)$$

$$+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)]$$

$$+\frac{2}{k_nL}\left[d_n^{(2)}(0)-B'_n(0)\right][a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)$$

$$+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)]$$

$$+\frac{2}{k_nL}\left[d_n^{(3)}(0)-B''_n(0)\right]a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)$$

$$-a_5\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$-a_5\frac{2}{k_nL}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du$$

$$+\int_0^t f_n(t-u)\,H(u-m\tau)\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du\}$$

Examples

We consider the flow of an Oldroyd-B fluid when the body force and the pressure gradient are omitted and the plate is accelerating. We present the analytical solution in the different initial conditions

Example 1 . In this example the plate is moving at speed ct, where c is constant. The corresponding initial problem is then given as

Separating variables and use of the Laplace transformation yields,

$$\bar{B}_n(s) = e^{sm\tau} \sum_{m=0}^{\infty} \sum_{k,i,j\geq 0}^{k+i+j=m} \frac{(-1)^m}{(\theta)^{m+1}} \frac{m!\left(-k_n{}^2\nu\right)^{j+l} \lambda_\beta{}^l \lambda_\alpha{}^i}{k!i!j!l!}$$

$$\{B_n(0)\left[\frac{s^{k+\alpha i+\beta l} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} + \lambda_\alpha \frac{s^{k+\alpha(i+1)+\beta l-1} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} + \theta \frac{s^{k+\alpha(i+2)+\beta l-1} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right.$$

$$\left. - \nu\lambda_\beta k_n{}^2 \frac{s^{k+\alpha i+\beta(1+l)-1} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right] + B'_n(0)\left[\lambda_\alpha \frac{s^{k+\alpha(1+i)+\beta l-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} + \theta \frac{s^{k+\alpha(i+2)+\beta l-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right.$$

$$\left. + \nu\lambda_\beta k_n{}^2 \frac{s^{k+\alpha i+\beta(1+l)-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right] - Me^{-s\tau} G(s) \frac{s^{k+\alpha i+\beta l} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}$$

$$+ \frac{2c}{k_n L}\left[\frac{s^{k+\alpha i+\beta l-1} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} - Me^{-s\tau} \frac{s^{k+\alpha i+\beta l-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} + Me^{-s\tau} H(s) \frac{s^{k+\alpha i+\beta l} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}} - \lambda_\alpha \frac{s^{k+\alpha(1+i)+\beta l-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right.$$

$$\left. - \theta\Gamma(-\alpha+2) \frac{s^{k+\alpha(2+i)+\beta l-2} e^{-sm\tau}}{\left(s^{2\alpha} - \frac{M}{\theta} e^{-s\tau}\right)^{m+1}}\right]\}$$

Taking inverse Laplace transform gives us

$$B_n(t) = \sum_{m=0}^{\infty} \sum_{k,i,j\geq 0}^{k+i+j=m} \frac{(-1)^m}{(\theta)^{m+1}} \frac{m!\left(-k_n{}^2\nu\right)^{j+l} \lambda_\beta{}^l \lambda_\alpha{}^i}{k!i!j!l!}$$

$$\{B_n(0) H(t-m\tau)\left[G_{2\alpha,-k-\alpha(i-2)-\beta l}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) + \lambda_\alpha G_{2\alpha,-k-\alpha(i-1)-\beta l+1}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) +\right.$$

$$\left. + \theta G_{2\alpha,-k-\alpha i-\beta l+1}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) - \nu\lambda_\beta k_n{}^2 G_{2\alpha,-k-\alpha(i-2)-\beta(1+l)+1}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau)\right]$$

$$+ B'_n(0)\left[\lambda_\alpha G_{2\alpha,-k-\alpha(i-1)-\beta l+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) + \theta G_{2\alpha,-k-\alpha i-\beta l+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau)\right.$$

$$\left. + \nu\lambda_\beta k_n{}^2 G_{2\alpha,-k-\alpha(i-2)-\beta(l+1)+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau)\right]$$

$$- M\int_0^t g(t-u) H(u-\tau(m+1)) G_{2\alpha,-k-\alpha(i-2)-\beta l}^{\left(\frac{M}{\theta}\right),\tau,m}(u-\tau(m+1))\, du$$

$$+ \frac{2c}{k_n L} H(t-m\tau)\left[G_{2\alpha,-k-\alpha(i-2)-\beta l+1}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) - \lambda_\alpha G_{2\alpha,-k-\alpha(i-1)-\beta l+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau) -\right.$$

$$\left. \theta\Gamma(-\alpha+2) G_{2\alpha,-k-\alpha i-\beta l+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-m\tau)\right]$$

$$+ \frac{2cM}{k_n L} H(t-\tau(m+1)) G_{2\alpha,-k-\alpha(i-2)-\beta l+2}^{\left(\frac{M}{\theta}\right),\tau,m}(t-\tau(m+1))$$

$$+ \frac{2c}{k_n L}\int_0^t h(t-u) H(u-\tau(m+1)) G_{2\alpha,-k-\alpha(i-2)-\beta l}^{\left(\frac{M}{\theta}\right),\tau,m}(u-\tau(m+1))\, du\}$$

Example 2 . We consider the flow of an Oldroyd-B fluid with the initial conditions $\psi_1(y) = c$,$\psi_2(y) = 0$ and boundary conditions, $\varphi_1(t) = ct, \varphi_2(t) = 0$ where c is a constant. The problem now becomes,

$$u(y,t)=c,\qquad u(0,t)=ct,\qquad -\tau\le t\le 0,\ y>0,\ u_t(y,t)=0,\qquad u(L,t)=0,\qquad 0<\alpha,\beta<\tfrac{1}{2}.$$

Using the preceding method we obtain,

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j,q\ge 0}^{k+i+j+q=m}\frac{(-1)^m}{(M\lambda_\alpha)^{m+1}}\frac{m!\left(-k_n{}^2\nu\right)^{i+j}\lambda_\beta{}^j}{k!i!j!}$$
$$\{B_n(0)\,H(t-m\tau)\,[G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(t-m\tau)+\lambda_\alpha G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,-k-\beta q+1}(t-m\tau)$$
$$-\nu\lambda_\beta k_n{}^2 G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta(j+1)+1}(t-m\tau)]\ -M\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(u-\tau(m+1))\,du$$
$$+\frac{2c}{k_nL}H(t-m\tau)\left[G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j+1}(t-m\tau)-\lambda_\alpha G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,-k-\beta j+2}(t-m\tau)\right]$$
$$+\frac{2cM}{k_nL}H(t-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j+2}(t-\tau(m+1)$$
$$+\ \frac{2c}{k_nL}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(t-m\tau)\,du\}$$

after which $W(y,t)$ and so $u(y,t)$ may be found.

Conclusion

In this paper we used a variant of the method of separation of variables to simplify the governing fractional-order partial differential equations of a generalized viscoelastic Oldroyd-B fluid with constant delay in time to a set of fractional-order ordinary differential equations with homogeneous boundary condition. The Laplace transformation (followed by its inverse) was then employed to obtain the exact solutions of the linear fractional ordinary differential equation. The solutions are given in terms of multivariate Green functions. We found exact solutions for three specific situations illustrated by examples.

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